Mean-square Stability Analysis of an Extended Euler-Maruyama Method for a System of Stochastic Differential Equations

Ram Sharan Adhikari
Assistant Professor Of Mathematics
Rogers State University

Mathematical Finance, Probability, and Partial Differential Equation Conference
Rutgers University, New Brunswick, May 17-19, 2017

Table of contents

1. Introduction
2. The Extended Euler-Maruyama Method
3. Main Results
4. Examples
5. Linear Stability Analysis

Introduction

- Consider the system of stochastic differential equations

$$X(s) = x + \int_0^s b\left(X(r)\right) dr + \sum_{k=1}^{M} \int_0^s \sigma_k(X(r)\nu_k \mathrm{d}W_k(r), \\ X(0) = x \in \mathbb{R}^d, \tag{1.1}$$

- where $b : \mathbb{R}^d \to \mathbb{R}^d$, $\sigma_k : \mathbb{R}^d \to \mathbb{R}$, $\nu_k \in \mathbb{R}^d$ and $W_k(t)$ are independent, one dimensional Brownian motions.
- Assume b and σ_k are such that a weak solution to (1.1) exists and is unique in probability law.
- We would like to construct an accurate approximations to solutions of (1.1) on fixed time intervals.

Introduction

- SDEs have wide range of applications:
  - Population dynamics, protein kinetics and genetics, psychology and neuronal activity,
  - Math finance, turbulent diffusion and radio-astronomy,
  - Seismology and structural mechanics and so on.
- As with most ordinary differential equations we can not generally solve SDEs explicitly.
- Two ways of approximating solution to SDE:
  - Strong approximation.
  - Weak approximation.

Introduction

- We may be interested to approximate $\mathbb{E}(g(X_T))$, $T > 0$. For instance $\mathbb{E}\Big(e^{-rT}(X(T) - K)^+\Big)$.
- Better approximation on the probability distribution of solutions is sufficient.

Definition

Weak convergence of order β

$$\Big|\mathbb{E}f(X(T)) - \mathbb{E}f(Y(N))\Big| \le Kh^\beta, N = \frac{T}{h}.$$

The Numerical Method

- Literature review
[Anderson and Mattingly, 2011],[Bruti-Liberati and Platen, 2008], [Gaines and Lyons, 1997],[Kloeden and Platen, 1992].
- We propose a numerical method to approximate the solution of (1.1) with order of convergence one.
- Let $\pi := \{t_0 = 0 < t_1 < ... < t_N = T\}$ be a subdivision of $[0, T]$.
- Consider the same initial condition $X(0) = Y_0 = x_0$.
- $\left\{\eta_{1k}^{(i)}, \eta_{2k}^{(i)} : i \in \mathbb{N}, k \in \{1, 2, ...M\}\right\}$ is a collection of mutually independent normal random variables with mean zero and variance 1.
- Define $[a]^+ = \max\{a, 0\} = a \vee 0$ for all $a \in \mathbb{R}$.

The Numerical Method

- Fix $\theta \in (0,1)$ and define,

$$\alpha_1 = \frac{1}{2\theta(1-\theta)} \text{ and } \alpha_2 = \alpha_1 - 1 = \frac{1-2\theta+2\theta^2}{2\theta(1-\theta)}.$$

- Discretization step $h > 0$.
- For $i = 1, 2, 3, \ldots,$

$$\textbf{Step 1}: Y_1^* = Y_{i-1} + b(Y_{i-1})\theta h + \sum_{k=1}^{M} \sigma_k(Y_{i-1})\, \nu_k\, \eta_{1k}^i \sqrt{\theta h},$$

$$\textbf{Step 2}: Y_i = Y_1^* + b(Y_{i-1})(1-\theta)h$$

$$+ \sum_{k=1}^{M} \sqrt{[\alpha_1 \sigma_k^2(Y_1^*) - \alpha_2 \sigma_k^2(Y_{i-1})]^+}\, \nu_k\, \eta_{2k}^i \sqrt{(1-\theta)h}.$$

Motivation Behind The Algorithm

- Equation (1.1) is distributionally equivalent to

$$X(t) = X(0) + \int_0^t b\,(X(s))\,ds + \sum_{k=1}^{M} \nu_k \int_0^\infty \int_0^t I_{[0,\sigma_k^2(X(s)))}(u)\xi_k(du \times ds). \quad (2.1)$$

- ξ_k are independent space-time white noise processes.
- Must approximate $\xi_k(A_{[0,h)}(\sigma_k^2))$ to approximate the diffusion term in (2.1) over the interval $[0, h)$.
- $A_{[0,h)}(\sigma_k^2)$ is the region under the curve $\sigma_k^2(X(t))$ for $0 \le t \le h$.

Motivation Behind The Algorithm

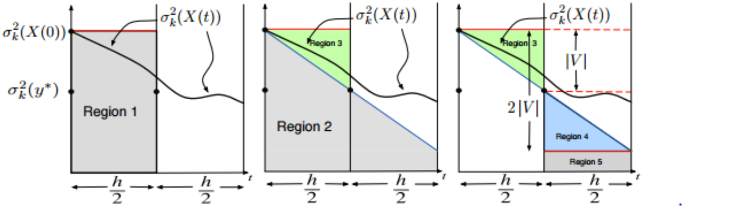

Figure: 1 A graphical illustration of Extended Euler-Maruyama Scheme for $\theta = \frac{1}{2}$

- To approximate $X(h)$, we focus on the double integral in (2.1) for a single k.

Motivation Behind The Algorithm

- Take $\theta = \frac{1}{2}$ for illustration.
- Approximate $X(\frac{h}{2})$ using the Euler-Maruyama method on the interval $[0, \frac{h}{2})$
- Denote it by Y_1^*
- To do so, we calculate ξ_k(Region 1), where Region 1 is the grey shaded region in the figure.

$$\xi_k(\text{Region } 1) \stackrel{d}{=} N(0, \sigma_k^2(X(0))\frac{h}{2}) \stackrel{d}{=} \sigma_k(X(0))\sqrt{\frac{h}{2}}N(0,1)$$

- This is equivalent in distribution to step 1 of our algorithm and exactly carried by step 1 of our algorithm.

Motivation Behind The Algorithm

- Consider the whole region Region 5.

$$\xi_k(\text{Region 5}) \stackrel{d}{=} N\Big(0, (\sigma_k^2(X(0)) - 2(\sigma_k^2(X(0)) - \sigma_k^2(Y_1^*))\frac{h}{2}\Big)$$
$$\stackrel{d}{=} N\Big(0, (2\sigma_k^2(Y_1^*) - \sigma_k^2(X(0)))\frac{h}{2}\Big)$$

- Returning to step 2 of our algorithm: For $i = 1$, we have $\sqrt{(2\sigma_k^2(Y_1^*)) - \sigma_k^2(X(0))}\sqrt{\frac{h}{2}}N(0,1)$ in step 2 of our algorithm.

Functional Setting

- We define $C^k(\mathbb{R}^d)$ in the following way:

$$C^k(\mathbb{R}^d) = \{f : \mathbb{R}^d \to \mathbb{R} \text{ s.t. } D^\alpha f(x) \text{ exists, bounded and continuous for all } x \in \mathbb{R}^d\}, \quad (3.1)$$

- where α is such that $\alpha_i \in \mathbb{Z}^+ \cup \{0\}$ for each $i = 1, \dots, d$ and $|\alpha| \le k$.
- We define the norm

$$||f||_k = \sup\left\{|D^\alpha f(x)| : x \in \mathbb{R}^d,\ \alpha = (\alpha_1, \dots, \alpha_d), |\alpha| \le k\right\}. \quad (3.2)$$

- $\mathcal{B}(\mathbb{R}^d)$ is the family of real-valued, bounded and Borel measurable functions defined on $\mathbb{R}^d$.

Functional Setting

- Define the Markov semigroup $\mathcal{P}_t : \mathcal{B}(\mathbb{R}^d) \to \mathcal{B}(\mathbb{R}^d)$ related to (1.1) by

$$(\mathcal{P}_t f)(x) :\overset{def}{=} \mathbb{E}_x f(X(t)), \text{ where } X(0) = x \tag{3.3}$$

- The Markov semigroup $P_h : \mathcal{B}(\mathbb{R}^d) \to \mathcal{B}(\mathbb{R}^d)$ by

$$(P_h f)(y) :\overset{def}{=} \mathbb{E}_y f(Y_1), \text{ where } Y_0 = y \tag{3.4}$$

.

Assumptions

(A1) The coefficients of (1.1) satisfy the Lipschitz and linear growth conditions:

$$
\begin{aligned}
&|b(x) - b(y)| + \sum_{k=1}^{M} |\sigma_k(x) - \sigma_k(y)| \le \kappa |x - y|, \\
&|b(x)| + \sum_{k=1}^{M} |\sigma_k(x)| \le \kappa (1 + |x|),
\end{aligned}
\tag{3.5}
$$

for all $k = 1, \ldots, M$ and $x, y \in \mathbb{R}^d$, where κ is a positive constant.

Assumptions

(A2) For each $k = 1, \dots, M$, we have $\inf_{x\in\mathbb{R}^d}\{\sigma_k(x)\} > 0$. In addition, there exists a positive constant $\lambda \in (0, 1]$ so that for any $x, \xi \in \mathbb{R}^d$ we have

$$\lambda|\xi|^2 \le \xi^T a(x)\xi \le \lambda^{-1}|\xi|^2, \tag{3.6}$$

where ξ^T denotes the transpose of ξ and $a(x) := \sum_{k=1}^M \sigma_k^2(x)\nu_k\nu_k^T$.

(A3) For all multi-index α with $|\alpha| \le 4$, we have

$$|D^\alpha b(x)| + \sum_{k=1}^{M}|D^\alpha\sigma_k(x)| \le K(1 + |x|^p), \text{ for all } x \in \mathbb{R}^d, \tag{3.7}$$

where K and p are positive numbers.

The Generators

$$(Af)(x) = f'[b(x)](x) + \frac{1}{2}\sum_{k=1}^{M}\sigma_k^2(x)f''[\nu_k,\nu_k](x), \qquad (3.8)$$

$$(B_1 f)(x) = f'[b(x_0)](x) + \frac{1}{2}\sum_{k=1}^{M}\sigma_k(x_0)^2 f''[\nu_k,\nu_k](x), \qquad (3.9)$$

$$(Bf)(x) = f'[b(x_0)](x) \qquad (3.10)$$
$$+ \frac{1}{2}\sum_{k=1}^{M}[\alpha_1\sigma_k^2(Y_1^*) - \alpha_2\sigma_k^2(x_0)]^+ f''[\nu_k,\nu_k](x).$$

- Where $f'[\nu](z)$ is the derivative of f in the direction ν evaluated at the point z.

Main Theorems

Theorem (Local Approximation)

Assume (A1)–(A3)*. Then there exist a constant K so that $\|\mathcal{P}_h - P_h\|_{4\to 0} \le Kh^2$ for all $h > 0$ sufficiently small.*

Theorem (Global Approximation)

Assume that $b \in C^4$ and $\forall k, \sigma_k \in C^4$ with $\inf_x \sigma_k(x) > 0$. Then for any $T > 0$ there exists a constant $C(T)$ such that

$$\sup_{0\le nh\le T} \|\mathcal{P}_{nh} - P_h^n\|_{4\to 0} \le C(T)h$$

Main Theorems

- The proof of the Local Approximation Theorem depends on the following lemma

Lemma

Assume (A1)–(A4). *Then for all $h > 0$ sufficiently small and $f \in C^4$, we have*

$$\mathbb{E}[f(Y_1^*) + (Bf)(Y_1^*)(1-\theta)h] = f(x_0) + (Af)(x_0)h + O(h^2).$$

Example 1 Geometric Brownian Motion

- For one dimensional case we have the Black-Scholes model.

$$\begin{aligned} dX(t) &= \lambda X(t)dt + \mu X(t)dW(t), \\ X(0) &= x \in \mathbb{R} \end{aligned} \tag{4.1}$$

- This Stochastic differential equation (4.1) is used to model asset price in financial mathematics.
- The solution to this SDE is

$$X(t) = X(0)e^{(\lambda - \frac{1}{2}\mu^2)t + \mu W(t)} \tag{4.2}$$

- The expected value of the solution is

$$\mathbb{E}[X(t)] = \mathbb{E}[X(0)]e^{\lambda t} \tag{4.3}$$

Example 1 Geometric Brownian Motion

- We test the weak convergence of our method over $[0, 1]$ for $\lambda = 2$, $\mu = 0.1$, and $X(0) = 1$.
- We used step sizes $h_p = 2^{p-10}$, for $1 \leq p \leq 5$ to sample over 50000 discretized brownian paths of the equation (4.1).
- We then compute the error and the outcome of the numerical experiment is shown in Figure 2 where we have plotted the weak error against 'h' on log-log scale.

Example 1 Geometric Brownian Motion

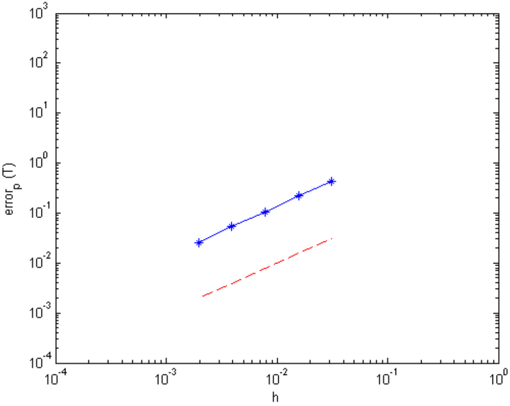

Figure: 2 Log-Log plots of error versus step-size.

Example 1 Geometric Brownian Motion

- Detailed calculation shows that

$$\mathbb{E}[(X(t))^2] = \mathbb{E}[X(0)^2]e^{(2\lambda+\mu^2)t} \tag{4.4}$$

- We test the weak convergence of our method over $[0, 1]$ for $\lambda = 2$, $\mu = 0.1$, and $X(0) = 1$.
- We used step sizes $h_p = 2^{p-10}$, for $1 \leq p \leq 5$ to sample over 50000 discretized brownian paths of the equation (4.1).
- The result of the numerical experiment is shown in the Figure 3 where we have plotted the weak error against 'h' on log-log scale.

Examples 1 Geometric Brownian Motion

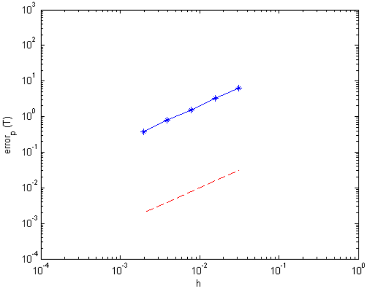

Figure: 3 Log-Log plots of error versus step-size.

- It is observed that the slope of the best fit line is empirically one.

Background For Linear Stability Analysis

- Roundoff and truncation errors produced during simulation is natural.
- The spreading of such errors is absolutely necessary to understand in simulation.
- The usefulness of a numerical method depends upon its ability to control the propogation of errors.
- Numerical schemes with better numerical stability are demanded.

Background For Linear Stability Analysis

- The solutions of a differential equation are continuous in their initial values, at least over a finite time interval.
- The concept of stability is an extension of this idea to an infinite time interval, [Kloeden and Platen, 1992].
- In some sense, the stability of a numerical scheme refers to the conditions under which the impact of an error vanishes asymptotically over time, [Bruti-Liberati and Platen, 2008].
- Consider scalar Itô equation

$$dX(t) = b(t, X(t))dt + \sigma(t, X(t))dW(t), X(t_0) = x_0 \in \mathbb{R}^d \quad (5.1)$$

- With a steady solution $X(t) \equiv 0$, so $b(t, 0) = 0$ and $\sigma(t, 0) = 0$.

Background For Linear Stability Analysis

- We recall the definitions for *pth*-mean stability and asymptotically stable in *pth*-mean from [Kloeden and Platen, 1992].

Definition

Steady solution $X_t \equiv 0$ is called

- stable in *p*th-mean if for every $\epsilon > 0$ and $t_0 \geq 0 \; \exists \; \delta = \delta(t_0, \epsilon) > 0$ such that $\mathbb{E}[|X_t^{t_0,x_0}|^p] < \epsilon$ for all $t \geq t_0$ and $|x_0| < \delta$,
- asymptotically stable in *p*th- mean if it is stable in *p*th-mean and there exists a $\delta_0 = \delta_0(t_0) > 0$ such that

$$\lim_{t\to\infty} \mathbb{E}[|X_t^{t_0,x_0}|^p] = 0 \quad \text{for all} \quad |x_0| < \delta_0.$$

Background For Linear Stability Analysis

- We give the following probabilistic definitions for stochastic stability from [Kloeden and Platen, 1992]

Definition

The steady solution $X(t) \equiv 0$ is called stochastically stable if for any $\epsilon > 0$ and $t_0 \geq 0$

$$\lim_{x_0 \to 0} \mathbb{P}\Big(\sup_{t \geq t_0} |X_t^{t_0,x_0}| \geq \epsilon \Big) = 0,$$

and stochastically asymptotically stable if, in addition,

$$\lim_{x_0 \to 0} \mathbb{P}\Big(\lim_{t \to \infty} |X_t^{t_0,x_0}| \to 0 \Big) = 1.$$

Background For Linear Stability Analysis

- Now it is natural to ask the following questions:
  1. Do the numerical solutions of SDEs preserve the stability properties of the original SDEs?
  2. and if yes for what stepsizes h does the numerical method reproduce the characteristics of the test equation?
- The above two questions have received quite a lot of attention and in fact stability analysis of numerical methods is motivated by them.
- We try to answer the above two questions.
- We will focus on the linear test equation

$$X(t) = X(0) + \int_0^t \lambda X(s)ds + \int_0^t \mu X(s)dW(s), \quad t \geq 0. \tag{5.2}$$

Mean-square Stability Analysis For Stochastic ODEs

- The SDE (5.2) can be written as

$$\begin{aligned} dX(t) &= \lambda(X(t))dt + \mu(X(t))dW(t), t \in [0, T] \\ X(0) &= x \in \mathbb{R} \end{aligned} \tag{5.3}$$

- We assume that λ and μ are real constants.
- Assuming that $X(0) \neq 0$ with probability 1, solutions of (5.3) have the following properties:

$$\lim_{t \to +\infty} \mathbb{E}(|X(t)|^2) = 0 \Leftrightarrow 2\lambda + \mu^2 < 0 \tag{5.4}$$

$$\lim_{t \to +\infty} |X(t)| = 0, \text{with probability 1} \Leftrightarrow \lambda - \frac{1}{2}\mu^2 < 0. \tag{5.5}$$

Here $\mathbb{E}(.)$ denotes the expected value.

Mean-square Stability Analysis For Stochastic ODEs

- For simplicity, let S_P denotes the sets of order pairs of real problem parameters for which problem is stable.
- Clearly,

$$S_P = \left\{\lambda, \mu \in \mathbb{R} : 2\lambda + \mu^2 < 0\right\}.$$

- Applying the extended Euler-Maruyama method to (5.2) produces the following iterative sequence:

$$Y_n = \left[A + B\eta_1^{(n)}\right] Y_{n-1} + \sqrt{(1-\theta)h\left[C + D\eta_1^{(n)} + E\left(\eta_1^{(n)}\right)^2\right]^+}\,\eta_2^{(n)}|Y_{n-1}|, \tag{5.6}$$

for $n = 1, 2, \ldots,$ where $\{\eta_1^{(n)}, \eta_2^{(n)}, n = 1, 2, \ldots\}$ are mutually independent normal random variables with mean zero and variance one.

Mean-square Stability Analysis For Stochastic ODEs

- A, B, C, D and E in (5.6) are given by

$$\begin{aligned} A &:= 1 + \lambda h, \\ B &:= \mu\sqrt{\theta h}, \\ C &:= \mu^2(1 + \alpha_1\lambda^2\theta^2 h^2 + 2\alpha_1\lambda\theta h) \\ D &:= 2\alpha_1\mu^3(\lambda\theta h + 1)\sqrt{\theta h}, \\ E &:= \alpha_1\mu^4\theta h > 0. \end{aligned} \tag{5.7}$$

- Apparently C is positive when $\lambda \geq 0$.
- When $\lambda < 0$, we can find $0 < h < \frac{1}{\lambda\theta}\left(-1 + \sqrt{\alpha_2/\alpha_1}\right)$ such that C is positive.

Mean-square Stability Analysis For Stochastic ODEs

- The sequence (5.6) is mean-square stable if $\lim_{n\to\infty}\mathbb{E}(|Y_n|^2)=0$) [Higham, 2000b].
- It follows from (5.6) that

$$\mathbb{E}[|Y_n|^2]=\mathbb{E}[|Y_{n-1}|^2]\Big(A^2+B^2+(1-\theta)h\mathbb{E}\Big[\Big[C+D\eta_1^{(n)}+E\big(\eta_1^{(n)}\big)^2\Big]^+\Big]\Big). \tag{5.8}$$

Lemma

For $0<h<\frac{1}{\lambda\theta}\big(-1+\sqrt{\alpha_2/\alpha_1}\big)$, *we have*

$$\mathbb{E}\Big[[C+D\eta_1^{(n)}+E(\eta_1^{(n)})^2]^+\Big]\le C+E+o(h^2). \tag{5.9}$$

Mean-square Stability Analysis For Stochastic ODEs

- Putting (5.9) into (5.8) and using the expressions for $A, \ldots, E$ in (5.7), detailed computations reveal that

$$\begin{aligned}\mathbb{E}[|Y_n|^2] &< \mathbb{E}[|Y_{n-1}|^2]\Big(A^2 + B^2 + (1-\theta)h(C + E + o(h^2))\Big)\\ &= \mathbb{E}[|Y_{n-1}|^2]\Big[1 + (2\lambda + \mu^2)h + \frac{1}{2}[2\lambda^2 + 2\lambda\mu^2 + \mu^4]h^2\\ &\qquad + o(h^2)\Big].\end{aligned} \tag{5.10}$$

- Furthermore, for the expression inside the brackets of the right-hand side of (5.10), we notice that

$$2\lambda^2 + 2\lambda\mu^2 + \mu^4 = 2\Big(\lambda + \frac{1}{2}\mu^2\Big)^2 + \frac{1}{4}\mu^4 > 0. \tag{5.11}$$

Mean-square Stability Analysis For Stochastic ODEs

- Next we compute the discriminant

$$\begin{aligned}\Delta &= (2\lambda + \mu^2)^2 - 4\frac{1}{2}(2\lambda^2 + 2\lambda\mu^2 + \mu^4) \\ &= -\mu^4 < 0.\end{aligned} \tag{5.12}$$

- Therefore, it follows that for any $h > 0$,

$$1 + (2\lambda + \mu^2)h + \frac{1}{2}[2\lambda^2 + 2\lambda\mu^2 + \mu^4]h^2 > 0.$$

- The condition for mean square stability of the Extended Euler-Maruyama(EEM) method for (5.2) is

$$1 + (2\lambda + \mu^2)h + \frac{1}{2}[2\lambda^2 + 2\lambda\mu^2 + \mu^4]h^2 < 1. \tag{5.13}$$

Mean-square Stability Analysis For Stochastic ODEs

- We can rewrite equation (5.13) as

$$0 < h < \frac{-2(2\lambda + \mu^2)}{2\lambda^2 + 2\lambda\mu^2 + \mu^4}. \tag{5.14}$$

- Sufficient condition for mean square stability of the weak Simpson method for (5.2) is

$$0 < h < \min\left\{\frac{-2(2\lambda + \mu^2)}{2\lambda^2 + 2\lambda\mu^2 + \mu^4}, \frac{1}{\lambda\theta}\left(-1 + \sqrt{\alpha_2/\alpha_1}\right)\right\}. \tag{5.15}$$

Mean-square Stability Analysis For Stochastic ODEs

- We have the following theorem:

Theorem

The following assertions are true:

(a) *Given* $(\lambda, \mu) \in S_P$*, the extended Euler-Maruyama method is mean-square stable if the discretization stepsize h satisfies* (5.15)*. Therefore the mean square stability of the process* (5.2) *implies the mean square stability of the extended Euler-Maruyama method if the discretization stepsize h satisfies* (5.15)*.*

(b) *Conversely, if the extended Euler-Maruyama method with discretization stepsize* $h > 0$ *is mean-square stable for* (5.2)*, then the parameters* λ *and* μ *of* (5.2) *satisfies* $(\lambda, \mu) \in S_P$*. In other words, the mean square stability of the extended Euler-Maruyama method implies that of the underlying stochastic process* (5.2)*.*

Mean-square Stability Analysis For Stochastic ODEs

- We can visualize the stability region for

$$\frac{-2(2\lambda+\mu^2)}{2\lambda^2+2\lambda\mu^2+\mu^4} < \frac{1}{\lambda\theta}\Big(-1+\sqrt{\alpha_2/\alpha_1}\Big).$$

- Using (5.13), we have

$$S_M := \Big\{(x,y) \in \mathbb{R}^2 : 2x^2+2xy+y^2+4x+2y<0\Big\}. \qquad (5.16)$$

- The plot of mean-square stability domain for the EEM method S_M is shown in Figure 9.

Mean-square Stability Analysis For Stochastic ODEs

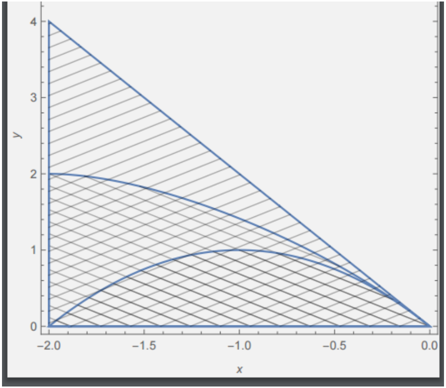

Figure: 9 Real mean-square stability domain for weak EEM and EM method

Thank You!

Anderson, D. F. and Mattingly, J. C. (2011).

A weak trapezoidal method for a class of stochastic differential equations.

Commun. Math. Sci., 9(1):301–318.

Asmussen, S. and Glynn, P. W. (2007).

Stochastic simulation: algorithms and analysis, volume 57 of *Stochastic Modelling and Applied Probability*.

Springer, New York.

Atkinson, K. E. (1989).

An introduction to numerical analysis.

John Wiley & Sons, Inc., New York, second edition.

Bruti-Liberati, N. and Platen, E. (2008).

Strong predictor-corrector Euler methods for stochastic differential equations.

Stoch. Dyn., 8(3):561–581.

Ethier, S. N. and Kurtz, T. G. (1986).

Markov processes: Characterization and Convergence.

Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. John Wiley & Sons Inc., New York.

Gaines, J. G. and Lyons, T. J. (1997).

Variable step size control in the numerical solution of stochastic differential equations.

SIAM J. Appl. Math., 57(5):1455–1484.

Higham, D. J. (2000a).

A-stability and stochastic mean-square stability.

BIT, 40(2):404–409.

Higham, D. J. (2000b).

Mean-square and asymptotic stability of the stochastic theta method.

SIAM J. Numer. Anal., 38(3):753–769 (electronic).

Higham, D. J. (2001).

An algorithmic introduction to numerical simulation of stochastic differential equations.

SIAM Rev., 43(3):525–546 (electronic).

Karatzas, I. and Shreve, S. E. (1991).

Brownian motion and stochastic calculus, volume 113 of *Graduate Texts in Mathematics*.

Springer-Verlag, New York, second edition.

Kloeden, P. E. and Platen, E. (1992).

Numerical solution of stochastic differential equations, volume 23 of *Applications of Mathematics (New York)*.

Springer-Verlag, Berlin.

Øksendal, B. (2003).

Stochastic differential equations, An introduction with applications.

Universitext. Springer-Verlag, Berlin, sixth edition.

Saito, Y. and Mitsui, T. (1996).

Stability analysis of numerical schemes for stochastic differential equations.

SIAM J. Numer. Anal., 33(6):2254–2267.

Saito, Y. and Mitsui, T. (2002).

Mean-square stability of numerical schemes for stochastic differential systems.

Vietnam Journal of Mathematics 30(SI):551-560.

Sauer, T. (2012).

Numerical solution of stochastic differential equations in finance.

In *Handbook of computational finance*, Springer Handb. Comput. Stat., pages 529–550. Springer, Heidelberg.

Talay, D. and Tubaro, L. (1990).

Expansion of the global error for numerical schemes solving stochastic differential equations.

Stochastic Anal. Appl., 8(4):483–509 (1991).

Steele, J. (2010).

Stochastic Calculus and Financial Applications.

Springer-Verlag, Newyork.

Khasminskii, R. (2012).

Stochastic stability of differential equations, volume 66 of *Stochastic Modelling and Applied probability,with contributions by G.N. Milstein and M.B. Nevelson*.

Heidlberg,second edition .

Mao, X. (1997).

Stochastic differential equations and application.

Horwood, Chichester .