Asset pricing under optimal contracts

Jakša Cvitanić (Caltech)

joint work with

Hao Xing (LSE)

Motivation and overview

- Existing literature:
  either
  - Prices are fixed, optimal contract is found
  or
  - Contract is fixed, prices are found in equilibrium
- An exception: Buffa-Vayanos-Woolley 2014 [BVW 14]
- However, [BVW 14] still severely restrict the set of admissible contracts
- We allow more general contracts and explore equilibrium implications

Literature

- Fixed contracts:

  Brennan (1993)
  Cuoco-Kaniel (2011)
  He-Krishnamurthy (2011)
  Lioui and Poncet (2013)
  Basak-Pavlova (2013)

- Fixed prices:

  Sung (1995)
  Ou-Yang (2003)
  Cadenillas, Cvitanić and Zapatero (2007)
  Leung (2014)
  Cvitanić, Possamai and Touzi, CPT (2016, 2017)

Buffa-Vayanos-Woolley 2014 [BVW 14]

- Optimal contract is obtained within the class

$$\text{compensation rate} = \phi \times \ \text{portfolio return} - \chi \times \ \text{index return}.$$

Our questions:

1. What is the optimal contract when investors are allowed to optimize in a larger class of contracts?
   (Linear contract is optimal in [Holmstrom-Milgrom 1987])

2. What are the equilibrium properties?

As shown in CPT (2016, 2017) ...

- The optimal contract depends on the output, its **quadratic variation**, the contractible sources of risk (if any), and the **cross-variations** between the output and the risk sources.

Our results

- Computing the optimal contract and equilibrium prices
- Optimal contract rewards Agent for taking specific risks and not only the systematic risk
- Stocks in large supply have high risk premia, while stocks in low supply have low risk premia
- Equilibrium asset prices distorted to a lesser extent:

  Second order sensitivity to agency frictions compared to the first order sensitivity in [BVW 14].

Outline

Assets

Riskless asset has an exogenous constant risk-free rate r.

Prices of N risky assets will be determined in equilibrium.

Dividend of asset i is given by

$$D_{it} = a_i p_t + e_{it},$$

where p and e_i follow Ornstein-Uhlenbeck processes

$$\begin{aligned} dp_t &= \kappa^p(\bar{p} - p_t)dt + \sigma_p dB_t^p, \\ de_{it} &= \kappa_i^e(\bar{e}_i - e_{it})dt + \sigma_{ei} dB_{it}^e. \end{aligned}$$

Vector of asset excess returns per share

$$dR_t = D_t dt + dS_t - rS_t dt.$$

The excess return of index

$$I_t = \eta' R_t,$$

where $\eta = (\eta_1, \ldots, \eta_N)'$ are the numbers of shares of assets in the market.

Available shares

Number of shares available to trade:
$\theta = (\theta_1, \ldots, \theta_N)'$
(Some assets may be held by buy-and-hold investors.)

We assume that η and θ are not linearly dependent. (Manager provides value to Investor.)

Portfolio manager

Portfolio manager's wealth process follows

$$d\bar{W}_t = r\bar{W}_t dt + (b\, m_t - \bar{c}_t)dt + dF_t,$$

- $\bar{c}_t$ is Manager's consumption rate
- F_t is the cumulative compensation paid by Investor
- $b\, m_t$ is the private benefit from his shirking action m_t, $b \in [0, 1]$, [DeMarzo-Sannikov 2006]
- No private investment
- Chooses portfolio Y for Investor

Investor

The reported portfolio value process:

$$G = \int_0^{\cdot} (Y_s' dR_s - m_s ds).$$

Investor observes only G and I

Her wealth process follows

$$dW_t = rW_t dt + dG_t + y_t dI_t - c_t dt - dF_t,$$

- Y_t is the vector of the numbers of shares chosen by Manager
- y_t is the number of shares of index chosen by Investor
- c_t is Investor's consumption rate
- m_t is Manager's shirking action, assumed to be nonnegative

Manager's optimization problem

Manager maximizes utility over intertemporal consumption:

$$\bar{V} = \max_{\bar{c},m,Y} \mathbb{E}\Big[\int_0^\infty e^{-\bar{\delta}t} u_A(\bar{c}_t)dt\Big],$$

- $\bar{\delta}$ is Manager's discounting rate
- $u_A(\bar{c}) = -\frac{1}{\bar{\rho}} e^{-\bar{\rho}\bar{c}}$

Manager's optimization problem

Manager maximizes utility over intertemporal consumption:

$$\bar{V} = \max_{\bar{c},m,Y} \mathbb{E}\Big[\int_0^\infty e^{-\bar{\delta}t} u_A(\bar{c}_t)dt\Big],$$

- $\bar{\delta}$ is Manager's discounting rate
- $u_A(\bar{c}) = -\frac{1}{\bar{\rho}} e^{-\bar{\rho}\bar{c}}$

If Manager is not employed by Investor, he maximizes

$$\bar{V}^u = \max_{\bar{c}^u,Y^u} \mathbb{E}\Big[\int_0^\infty e^{-\bar{\delta}t} u_A(\bar{c}_t^u)dt\Big]$$

subject to budget constraint

$$d\bar{W}_t = r\bar{W}_t + Y_t^u dR_t - \bar{c}_t^u dt.$$

Manager takes the contact if $\bar{V} \geq \bar{V}^u$.

Investor's maximization problem

Investor maximizes utility over intertemporal consumption:

$$V = \max_{c,F,y} \mathbb{E}\Big[\int_0^\infty e^{-\delta t} u_P(c_t) dt\Big],$$

- δ is Investor's discounting rate
- $u_P(c) = -\frac{1}{\rho} e^{-\rho c}$

Investor's maximization problem

Investor maximizes utility over intertemporal consumption:

$$V = \max_{c,F,y} \mathbb{E}\Big[\int_0^\infty e^{-\delta t} u_P(c_t) dt \Big],$$

- δ is Investor's discounting rate
- $u_P(c) = -\frac{1}{\rho} e^{-\rho c}$

If Investor does not hire Manager, she maximizes

$$V^u = \max_{c^u, y^u} \mathbb{E}\Big[\int_0^\infty e^{-\delta t} u_P(c_t^u) dt \Big]$$

subject to budget constraint

$$dW_t = rW_t + y_t^u dI_t - c_t^u dt.$$

Investor hires Manager if $V \geq V^u$.

Equilibrium

A price process S, a contract F in a class of contracts $\mathbb{F}$, and an index investment y, form an equilibrium if

1. Given S, $(F, \mathbb{F})$, and y, Manager takes the contract, and $Y = \theta - y\,\eta$ solves Manager's optimization problem.

2. Given S, Investor hires Manager, and (F, y) solves Investor's optimization problem, and F is the optimal contract in $\mathbb{F}$.

Outline

Asset prices

There exists an equilibrium with asset prices $S_{it} = a_{0i} + a_{pi} p_t + a_{ei} e_{it}$ (assuming θ and η are not linearly dependent.)

Setting $a_p = (a_{p1}, \ldots, a_{pN})'$ and $a_e = diag\{a_{e1}, \ldots, a_{eN}\}$, we have

$$a_{pi} = \frac{a_i}{r + \kappa^p} \quad a_{ei} = \frac{1}{r + \kappa_i^e}, \quad i = 1, \ldots, N,$$

(assuming the matrix $\Sigma_R = a_p \sigma_p^2 a_p' + a_e' \sigma_E^2 a_e$ is invertible.)

Asset prices

There exists an equilibrium with asset prices $S_{it} = a_{0i} + a_{pi} p_t + a_{ei} e_{it}$ (assuming θ and η are not linearly dependent.)

Setting $a_p = (a_{p1}, \ldots, a_{pN})'$ and $a_e = diag\{a_{e1}, \ldots, a_{eN}\}$, we have

$$a_{pi} = \frac{a_i}{r + \kappa^p} \quad a_{ei} = \frac{1}{r + \kappa_i^e}, \quad i = 1, \ldots, N,$$

(assuming the matrix $\Sigma_R = a_p \sigma_p^2 a_p' + a_e' \sigma_E^2 a_e$ is invertible.)

Notation:

$$Var^\eta = \eta' \Sigma_R \eta, \quad Covar^{\theta,\eta} = \eta' \Sigma_R \theta,$$

CAPM beta of the fund portfolio: $\beta^\theta = \dfrac{Covar^{\theta,\eta}}{Var^\eta}.$

Asset Returns

Asset excess returns are

$$\mu - r = r\frac{\rho\bar{\rho}}{\rho+\bar{\rho}}\Sigma_R\theta + r\mathscr{D}_b\Sigma_R(\theta - \beta^\theta\eta),$$

where

$$\mathscr{D}_b = (\rho+\bar{\rho})\left(b - \frac{\rho}{\rho+\bar{\rho}}\right)_+^2.$$

- When $b \in [0, \frac{\rho}{\rho+\bar{\rho}}]$, the first best is obtained.
- When $\frac{\theta_i}{\eta_i} > \beta^\theta$, risk premium of asset i increases with b.

  When $\frac{\theta_i}{\eta_i} < \beta^\theta$, risk premium of asset i decreases with b.

Asset prices/returns

In [BVW 14], $\mathscr{D}_b$ is replaced by

$$\mathscr{D}_b^{BVW} = \bar{\rho}\left(b - \frac{\rho}{\rho + \bar{\rho}}\right)_+.$$

Note that

$$\mathscr{D}_b < \mathscr{D}_b^{BVW}, \qquad \text{for any } b \in (0,1).$$

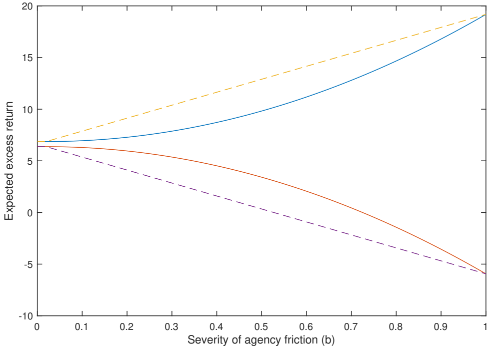


Figure: Solid lines: our result; Dashed lines: [BVW 14].

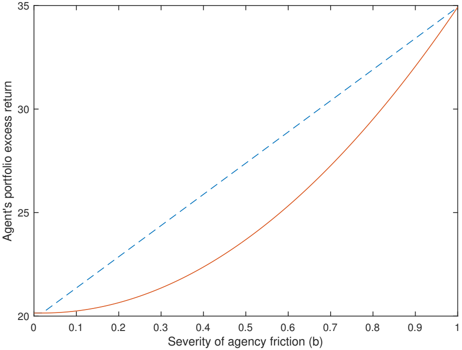
Index and portfolio returns

Excess return of the index

$$\eta'(\mu - r) = r\frac{\rho\bar{\rho}}{\rho + \bar{\rho}} Covar^{\theta,\eta}.$$

Excess return of Manager's portfolio

$$\theta'(\mu - r) = r\frac{\rho\bar{\rho}}{\rho + \bar{\rho}} Var^{\theta} + r\mathscr{D}_b\left(Var^{\theta} - \frac{(Covar^{\theta,\eta})^2}{Var^{\eta}}\right).$$



Optimal contract

$$dF_t = Cdt + \frac{\rho}{\rho+\bar{\rho}} dG_t + \xi(dG_t - \beta^\theta dI_t) + \frac{r}{2}\zeta\, d\langle G - \beta^\theta I, G^\theta - \beta^\theta I\rangle_t$$

- Optimality in a large class of contracts
- **Conjecture:** It is optimal in general.
- $\xi = (b - \frac{\rho}{\rho+\bar{\rho}})_+$, $\zeta = (\rho+\bar{\rho})(b+\xi)(1-b-\xi)\xi$
- When $b \leq \frac{\rho}{\rho+\bar{\rho}}$, $\xi = \zeta = 0$, only the first two terms show up. The return of the fund is shared between investor and portfolio manager with ratio $\frac{\rho}{\rho+\bar{\rho}}$.

BVW 14 contract corresponds to the two terms in the middle.

- The quadratic variation term is new.
- $\langle G - \beta^\theta I, G - \beta^\theta I\rangle$ can be thought as a *tracking gap*.

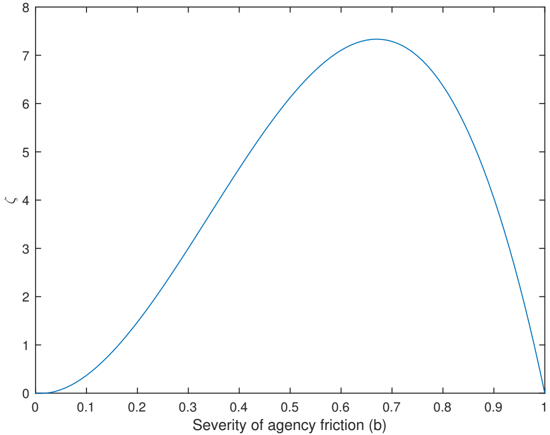
  Tracking gap is rewarded to motivate Manager to take the specific risk of individual stocks, and not only the systematic risk of the index.

Optimal contract

When $b \geq \frac{\rho}{\rho+\bar{\rho}}$,

ξ is increasing in b, so as to make Manager to not employ the shirking action.

Dependence of ζ on b:



New contract improves Investor's value

For the asset price in [BVW 14], Investor's value is improved by using the new contract.

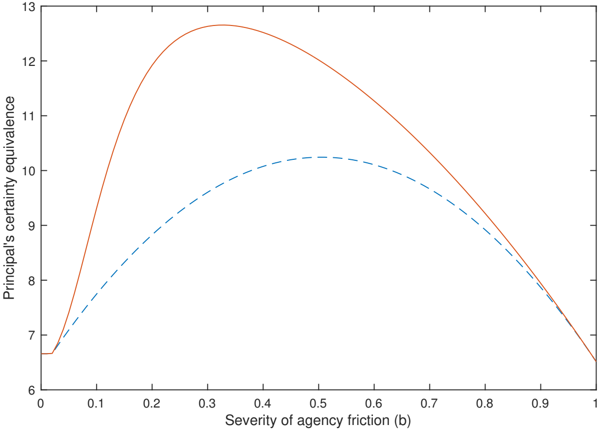


Figure: Solid line: our contract, Dashed line: [BVW 14]

Outline

Admissible contracts: motivation

For any Manager's admissible strategy $\Xi = (\bar{c}, Y, m)$, consider

$$\Xi^t = \{\hat{\Xi} \text{ admissible } \mid \hat{\Xi}_s = \Xi_s, s \in [0, t]\}.$$

Define Manager's *continuation value process* $\bar{\mathcal{V}}(\Xi)$ as

$$\bar{\mathcal{V}}_t(\Xi) = \text{ess sup}_{\Xi^t} \mathbb{E}_t\Big[\int_t^\infty e^{-\bar{\delta}(s-t)} u_A(\bar{c}_s) ds\Big], \quad t \geq 0.$$

Admissible contracts: motivation

For any Manager's admissible strategy $\Xi = (\bar{c}, Y, m)$, consider

$$\Xi^t = \{\hat{\Xi} \text{ admissible } \mid \hat{\Xi}_s = \Xi_s, s \in [0, t]\}.$$

Define Manager's *continuation value process* $\bar{\mathcal{V}}(\Xi)$ as

$$\bar{\mathcal{V}}_t(\Xi) = \text{ess sup}_{\Xi^t} \mathbb{E}_t \Big[\int_t^\infty e^{-\bar{\delta}(s-t)} u_A(\bar{c}_s) ds \Big], \quad t \geq 0.$$

(i) $\partial_{\bar{W}_t} \bar{\mathcal{V}}_t(\Xi) = -r\bar{\rho}\bar{\mathcal{V}}_t(\Xi)$;

(ii) Transversality condition: $\lim_{t\to\infty} \mathbb{E}\big[e^{-\bar{\delta}t}\bar{\mathcal{V}}_t(\Xi)\big] = 0$;

(iii) Martingale principle:

$$\tilde{\mathcal{V}}_t(\Xi) = e^{-\bar{\delta}t}\bar{\mathcal{V}}_t(\Xi) + \int_0^t e^{-\bar{\delta}s} u_A(\bar{c}_s) ds,$$

is a supermartingale for arbitrary admissible strategy Ξ, and is a martingale for the optimal strategy Ξ^*.

Admissible contracts: definition

(Motivated by CPT (2016), (2017))

A contract F is admissible if

1. there exists a constant $\bar{V}_0$,
2. for any Agent's strategy there exist $\mathbb{F}^{G,I}$-adapted processes $Z, U, \Gamma^G, \Gamma^I, \Gamma^{GI}$ such that the process $\bar{V}(\Xi)$, defined via

$$\begin{aligned} d\bar{V}_t(\Xi) =& X_t\Big[(bm_t - \bar{c}_t)dt + Z_t dG_t + U_t dI_t \\ & + \tfrac{1}{2}\Gamma^G_t d\langle G, G\rangle_t + \tfrac{1}{2}\Gamma^I_t d\langle I, I\rangle_t + \Gamma^{GI}_t d\langle G, I\rangle_t\Big] \\ & + \bar{\delta}\bar{V}_t(\Xi)dt - H_t dt, \quad \bar{V}_0(\Xi) = \bar{V}_0, \end{aligned}$$

where $X_t = -r\bar{\rho}\bar{V}_t(\Xi)$ and H is the Hamiltonian

$$\begin{aligned} H = \sup_{\bar{c}, m\geq 0, Y} \Big\{ u_A(\bar{c}) + X\Big[bm - \bar{c} - Zm + ZY'(\mu - r) + U\eta'(\mu - r) \\ + \tfrac{1}{2}\Gamma^G Y'\Sigma_R Y + \tfrac{1}{2}\Gamma^I \eta'\Sigma_R \eta + \Gamma^{GI} Y'\Sigma_R \eta\Big]\Big\}, \end{aligned}$$

satisfies $\lim_{t\to\infty} \mathbb{E}\big[e^{-\bar{\delta}t}\bar{V}_t(\Xi)\big] = 0$.

Manager's optimal strategy

Lemma

Given an admissible contract with

$$X > 0, \quad Z \geq b, \quad \text{and} \quad \Gamma^G < 0,$$

the Manager's optimal strategy is the one maximizing the Hamiltonian,

$$\bar{c}^* = (u_A')^{-1}(X), \quad m^* = 0,$$

$$Y^* + y\eta = -\frac{Z}{\Gamma^G}\Sigma_R^{-1}(\mu - r) - \frac{\Gamma^{GI}}{\Gamma^G}\eta,$$

and we have

$$\bar{V}(\Xi) = \hat{\mathcal{V}}(\Xi).$$

Do we lose on generality?

[CPT 2016, 2016] considered the finite horizon case,

$$d\bar{V}_t = X_t\Big[bm_t dt + Z_t dG_t + U_t dI_t + \frac{1}{2}\Gamma_t^G d\langle G, G\rangle_t + \tfrac{1}{2}\Gamma_t^I d\langle I, I\rangle_t + \Gamma_t^{GI}\langle G, I\rangle_t\Big] - H_t dt.$$

$\bar{V}_T = \mathcal{C}_T$ is the lump-sum compensation paid.

They showed the set of $\mathcal{C}$ that can be represented as $\bar{V}_T$ is dense in the set of all (reasonable) contracts. Hence, there is no loss of generality in their framework.

Their proof is based on the 2BSDE theory, e.g., [Soner-Touzi-Zhang 2011,12,13].

Conjecture: A similar result holds for the infinite horizon case. (Work in progress by Lin, Ren, and Touzi.)

Representation of admissible contracts

Lemma

An admissible contract F can be represented as

$$\begin{aligned} dF_t =& Z_t dG_t + U_t dI_t + \tfrac{1}{2}\Gamma^G_t\, d\langle G, G\rangle_t + \tfrac{1}{2}\Gamma^I_t\, d\langle I, I\rangle_t + \Gamma^{GI}_t\, d\langle G, I\rangle_t \\ &+ \tfrac{1}{2} r\bar{\rho}\, d\langle Z\cdot G + U\cdot I, Z\cdot G + U\cdot I\rangle_t - \left(\tfrac{\bar{\delta}}{r\bar{\rho}} + \bar{H}_t\right)dt, \end{aligned}$$

where $Z\cdot G = \int_0^\cdot Z_s dG_s$ *and*

$$\begin{aligned} \bar{H}_t =& \tfrac{1}{\bar{\rho}}\log(-r\bar{\rho}\bar{V}_0) - \tfrac{1}{\bar{\rho}} + (Z_t Y^*_t + U_t\eta)'(\mu_t - r) \\ &+ \tfrac{1}{2}\Gamma^G_t\,(Y^*_t)'\Sigma_R Y^*_t + \tfrac{1}{2}\Gamma^I_t\,\eta'\Sigma_R\eta + \Gamma^{GI}_t\,(Y^*_t)'\Sigma_R\eta. \end{aligned}$$

In particular, F is adapted to $\mathbb{F}^{G,I}$ (as it should be).

Investor's problem

1. Guess Investor's value function

$$V(w) = Ke^{-r\rho w},$$

2. Treat $Z, U, \Gamma^{G}, \Gamma^{GI}$ as Investor's control variables.

3. Work the with HJB equation satisfied by V.

Conclusion

- We find an asset pricing equilibrium with the contract optimal in a large class. (Maybe the largest.)
- Price/return distortion less sensitive to agency frictions.
- The contract based on the "tracking gap" and its quadratic variation.

Future work:

- Square root, CIR dividend processes

Outline

Example: delegated portfolio management

The portfolio value process X_t follows the dynamics

$$dX = \frac{v_1}{S_1} dS_1 + \frac{v_2}{S_2} dS_2$$

where

$$dS_{i,t}/S_{i,t} = b_i dt + dB_t^i,$$

where B^i are independent Brownian motions and b_i are constants. We have then

$$dX_t = [v_{1,t} b_1 + v_{2,t} b_2] \, dt + v_{1,t} dB_t^1 + v_{2,t} dB_t^2.$$

The principal hires an agent to manage the values of $v_t = (v_{1,t}, v_{2,t})$.

Expected utility

The agent is paid at the final time T in the amount C_T, and draws expected utility

$$-E\left[e^{-R_A\left(C_T-\int_0^T c(v_1(s),v_2(s))ds\right)}\right]$$

where the agent's running cost is of the form

$$c(v_1,v_2)=\frac{1}{2}\beta_1(v_1-\alpha_1)^2+\frac{1}{2}\beta_2(v_2-\alpha_2)^2$$

The principal's expected utility is

$$-E\left[e^{-R_P(X_T-C_T)}\right]$$

First best

Given a "bargaining-power" parameter $\rho > 0$, the first-best (risk-sharing) problem is

$$\max_{v} \max_{C_T} E\left[U_P(X_T - C_T) + \rho U_A(C_T - \int_0^T c(v_1(s), v_2(s))ds) \right]$$

The first order condition for C_T is then

$$\frac{U_P'(X_T - C_T)}{U_A'(C_T - K_T^v)} = \rho$$

With CARA utilities, we obtain

$$C_T = \frac{1}{R_A + R_P}\left(R_P X_T + R_A K_T^v + \log\left(\frac{\rho R_A}{R_P}\right)\right)$$

Second best

We consider linear contracts based on the path of the observable portfolio value X, the observable quadratic variation of X, and, possibly, on S_1 via B^1, and the co-variation of X and B_1. Indicator $\mathbf{1}_O$ indicates whether S_1 is observed.

$$C_T = C_0 + \int_0^T \left[Z_s^X dX_s + Y_s^X d\langle X \rangle_s + \mathbf{1}_O \left(Z_s^1 dB_s^1 + Y_s^1 d\langle X, B_1 \rangle_s \right) + H_s ds \right], \tag{1}$$

for some constant C_0, and some adapted processes Z^X, Z^1, Y^X, Y^1 and H. Transformation of variables:

$$Y^X = \frac{1}{2}\left(\Gamma^X + R_A (Z^X)^2\right),$$
$$Y^1 = \Gamma^1 + R_A Z^X Z^1,$$
$$H = -G + \frac{1}{2} R_A (Z^1)^2.$$

SIMPLE CRUCIAL OBSERVATION: $C_T =$ agent's value function at T.
We will argue then that the natural choice for G_t is
$G_t := G(Z_t^X, Z_t^1, \Gamma_t^X, \Gamma_t^1)$, where

$$G(Z^X, Z^1, \Gamma^X, \Gamma^1) := \sup_{v_1, v_2} g(v_1, v_2, Z^X, Z^1, \Gamma^X, \Gamma^1)$$
$$:= \sup_{v_1, v_2} \left\{ \frac{1}{2}\Gamma^X(v_1^2 + v_2^2) + Z^X b \cdot v - c(v_1, v_2) + \mathbf{1}_O \Gamma^1 v_1 \right\}.$$

The agent is maximizing

$$-\mathbb{E}_t^{\mathbb{P}}\left[e^{-R_A \int_0^T [g_s - G_s] ds}\right] \leq 1,$$

Any pair $(v_1^*(s), v_2^*(s))$ that maximizes $g_s := g(Z_s^X, Z_s^1, \Gamma_s^X, \Gamma_s^1)$ is optimal.

Contractible S_1: first best is attained

Optimal (v_1^*, v_2^*) is obtained by maximizing

$$g = -\frac{1}{2}\beta_1(v_1 - \alpha_1)^2 - \frac{1}{2}\beta_2(v_2 - \alpha_2)^2$$

$$+Z^X b \cdot v + \Gamma^1 v_1 + \frac{1}{2}\Gamma^X \|v\|^2 + Z^1 b_1 + (Z^1)^2 + 2Z^1 Z^X v_1.$$

Assume, for example, that $b_2 \neq 0$, $\beta_2 \leq \beta_1$. Suppose the principal sets

$$\begin{aligned}
\Gamma_t^X &\equiv \beta_2, \\
Z_t^X &\equiv -\alpha_2\beta_2/b_2, \\
\Gamma_t^1 &= -\alpha_1\beta_1 - Z_t^X b_1 + (\beta_1 - \beta_2)v_1^{FB}, \\
Z_1 &\equiv 0.
\end{aligned}$$

Then,

$$g = (\beta_2 - \beta_1)\left[\frac{1}{2}v_1^2 - v_1 v_1^{FB}\right] + const.$$

Agent is indifferent with respect to v_2, and he chooses $v_1^* = v_1^{FB}$.

Non-contractible S_1

Optimal (v_1^*, v_2^*) is obtained by maximizing

$$g(v_1, v_2) = -\frac{1}{2}\beta_1(v_1 - \alpha_1)^2 - \frac{1}{2}\beta_2(v_2 - \alpha_2)^2 + Z^X b \cdot v + \frac{1}{2}\Gamma^X \|v\|^2.$$

Assume, for example, $\beta_2 \leq \beta_1$. If $\Gamma^X < \beta_2 \leq \beta_1$, the optimal positions are

$$v_i^* = \frac{Z^X b_i + \alpha_i \beta_i}{\beta_i - \Gamma^X}.$$

The principal maximizes, over Z and Γ,

$$b \cdot v^*(Z, \Gamma) - \frac{1}{2}[R_A Z^2 + R_P(1 - Z)^2]\|v^*\|^2 - c(v^*(Z, \Gamma)).$$

Main messages from numerics

- 1. The percentage loss in the principal's second best utility certainty equivalent relative to the first best, when varying initial risk expoosure α_2, can be significant for extreme values of α_2. That is, when the initial risk exposure is far from desirable, the moral hazard cost of providing incentives to the agent to modify the exposure is high.
- 2. The loss in the principal's second best certainty equivalent relative to the one she would obtain if offering the contract that is optimal among those that are linear in the output, but do not depend on its quadratic variation, can also be large.
- 3. The principal uses quadratic variation as an incentive tool: for low values of the initial risk exposure she wants to increase the risk exposure by rewarding higher variation, and for its high values she wants to decrease it by penalizing high variation.

Example: Quadratic cost, drift effort; C., Wan and Zhang (2009)

$$dX_t = \sigma\alpha_t dt + \sigma dB_t^\alpha$$

Agent is maximizing $E[U_A(C_T) - \frac{c}{2}\int_0^T \alpha_t^2 dt]$, while Principal is maximizing $E[U_P(X_T - C_T)]$

Proposition. Assuming that Principal's value function $V^P(t,x,y)$ is in class $C^{2,3,3}$, we have, for some constant L,

$$V_y^P(t, X_t, Y_t) = -\frac{1}{c}V^P(t, X_t, Y_t) - L$$

In particular, the optimal contract C_T satisfies

$$\frac{\tilde{U}_P'(X_T - C_T)}{U_A'(C_T)} = \frac{1}{c}U_P(X_T - C_T) + L \qquad (2)$$

Proof:

The HJB equation for Principal's value function $v(t,x,y) = V^P(t,x,y)$ is

$$\begin{cases} \partial_t v + \sup_z \left\{ \frac{1}{c}\sigma^2 z v_x + \frac{1}{2c}\sigma^2 z^2 v_y + \frac{1}{2}\sigma^2 \left(v_{\mathrm{xx}} + z^2 v_{\mathrm{yy}} \right) + \sigma^2 z v_{\mathrm{xy}} \right\} = 0, \\ v(T,\mathrm{x},y) = U_P(\mathrm{x} - U_A^{-1}(y)). \end{cases}$$

Optimizing over z gives

$$z^* = -\frac{v_x + c v_{xy}}{v_y + c v_{yy}}$$

We have that $v(t, X_t, Y_t)$ is a martingale under the optimal measure P, satisfying

$$dv = \sigma(v_x + z^* v_y) dW$$

Then, compare to dv_y, with boundary condition

$$v_y(T,\mathrm{x},y) = -\frac{U_P'(\mathrm{x} - U_A^{-1}(y))}{U_A'(U_A^{-1}(y))}$$

Risk-neutral principal and logarithmic agent

Suppose $c = 1$,

$$U_P(C_T) = X_T - C_T \ , \quad U_A(C_T) = \log C_T .$$

We also assume

$$dX_t = \sigma \alpha_t X_t dt + \sigma X_t dB_t^{\alpha} .$$

The optimal contract payoff C_T satisfies

$$C_T = \frac{1}{2} X_T + const.$$

Example: Risk-Sharing in complete markets Cadenillas, Cvitanić and Zapatero (2007)

Assume a complete market with no cost on choosing the "portfolio strategy". Using these methods we recover the result from the above paper that the optimal payoff $F(X_T)$ is given by solving the ODE

$$\frac{U_P'(x - F(x))}{U_A'(F(x))} = \kappa F'(x)$$

This gives a linear contract for CARA utilities.
Also for CRRA utilities, but only with the same risk aversions.

Thank you for your attention!

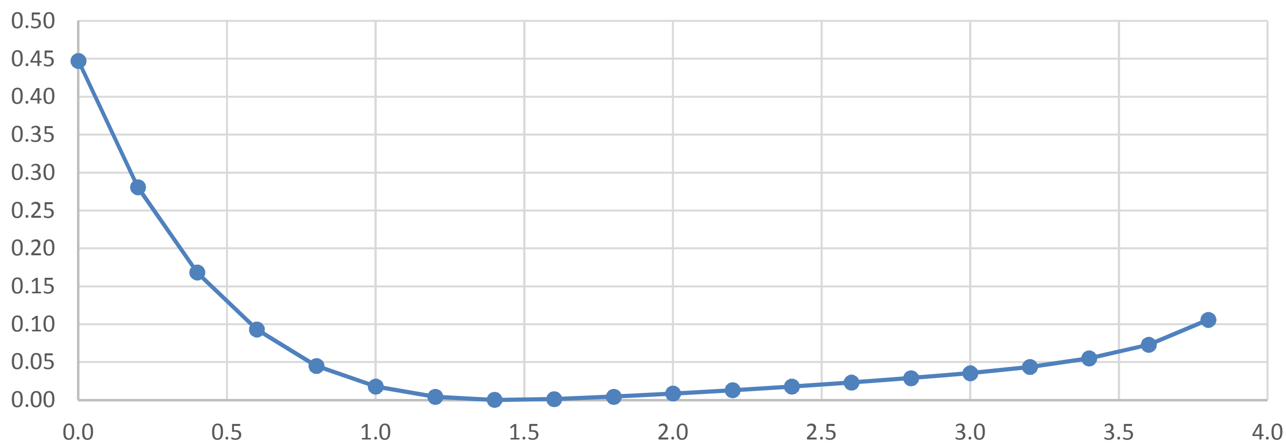


Figure 1: Percentage loss in principal's certainty equivalent relative to first best, as function of α_2.
Parameter values: R_A=10, R_P=0.58, α_1=0.5, β_1=0.4, β_2=1, b_1=0.4, b_2=1, B_0=0.

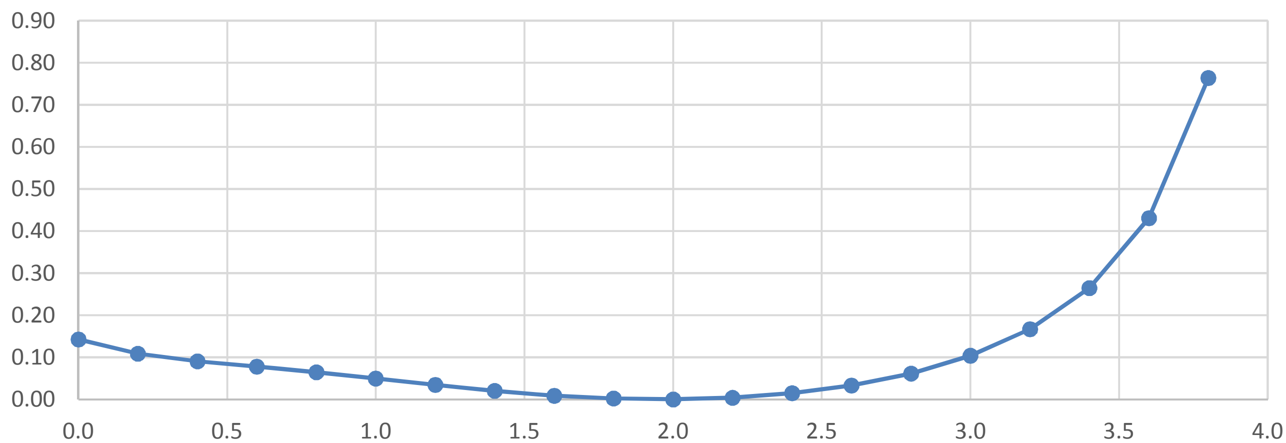


Figure 2: Percentage loss in principal's certainty equivalent when not using quadratic variation, as function of α_2.
Parameter values: R_A=10, R_P=0.58, α_1=0.5, β_1=0.4, β_2=1, b_1=0.4, b_2=1, B_0=0.

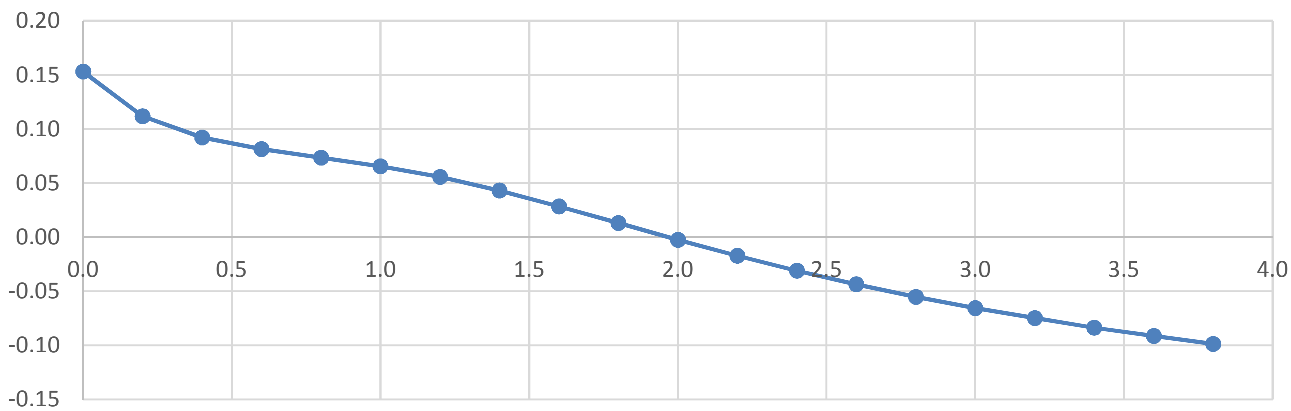


Figure 3: Optimal contract's sensitivity to quadratic variation, as function of α_2.
Parameter values: R_A=10, R_P=0.58, α_1=0.5, β_1=0.4, β_2=1, b_1=0.4, b_2=1, B_0=0.