Model Uncertainty, Recalibration, and the Emergence of Delta-Vega Hedging

Sebastian Herrmann
University of Michigan

Joint work with Johannes Muhle-Karbe

Conference on Mathematical Finance, Probability, and PDEs

Rutgers University, New Brunswick, May 17, 2017

Motivation

Hedging: Theory vs. Practice

- Hedging problem: Use traded assets to manage the risk associated with a short position in a derivative security.
- Classical mathematical finance:
  - Specify a stochastic model: describes *stochastic* behaviour of certain financial *variables* in terms of *deterministic* input quantities, the model's *parameters*.
  - Compute hedging strategy according to some criterion.
  - Example: *delta hedging* in the Black–Scholes model.

Motivation

Hedging: Theory vs. Practice

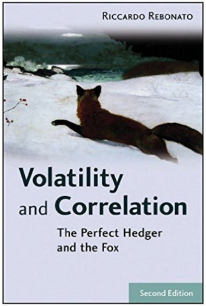
- Hedging problem: Use traded assets to manage the risk associated with a short position in a derivative security.
- Classical mathematical finance:
  - Specify a stochastic model: describes *stochastic* behaviour of certain financial *variables* in terms of *deterministic* input quantities, the model's *parameters*.
  - Compute hedging strategy according to some criterion.
  - Example: *delta hedging* in the Black–Scholes model.
- In practice:
  - Parameters are frequently **recalibrated** to market prices of traded options.
  - **Out-of-model hedging**: sensitivities with respect to changes in these parameters are hedged by trading in options.
  - Example: Recalibration of the Black–Scholes volatility parameter and *vega hedging*.
- Logically inconsistent.

Motivation

Out-of-model hedging

Rebonato ’05:

> *“Needless to say,* ***out-of-model hedging is on conceptually rather shaky ground****: if the volatility is deterministic and perfectly known, as many models used to arrive at the price assume it to be, there would be no need to undertake vega hedging. Furthermore, calculating the vega statistics means estimating the dependence on changes in volatility of a price that has been arrived at assuming the self-same volatility to be both deterministic and perfectly known. Despite these logical problems,* ***the adoption of out-of-model hedging in general, and of vega hedging in particular, is universal*** *in the complex-derivatives trading community.”*



This paper

- Hedging problem with dynamic trading in three liquid assets: stock, bond, and a “call” on the stock.
- Preferences: Moderate risk and uncertainty aversion around a recalibrated Black–Scholes model.
- Goal: Find almost optimal hedging strategies and indifference prices.
- To obtain explicit formulas, study limit for *small* uncertainty aversion.

This paper

- Hedging problem with dynamic trading in three liquid assets: stock, bond, and a “call” on the stock.
- Preferences: Moderate risk and uncertainty aversion around a recalibrated Black–Scholes model.
- Goal: Find almost optimal hedging strategies and indifference prices.
- To obtain explicit formulas, study limit for *small* uncertainty aversion.
- Delta-vega hedging arises naturally as an asymptotically optimal strategy.
- Asymptotic indifference price corrections are determined by disparity between vegas and second-order greeks (gammas, vannas, volgas) of the non-traded option and the call.

Outline

Model Uncertainty

Hedging and models

- Crucial for hedging: accurate description of future volatility of the stock, i.e., a *stochastic model* for the financial market.
- Prime example: Black–Scholes model.
  - Input: volatility parameter.
  - Output: unique arbitrage-free price and replicating strategy for the option.

Hedging and models

- Crucial for hedging: accurate description of future volatility of the stock, i.e., a *stochastic model* for the financial market.
- Prime example: Black–Scholes model.
  - Input: volatility parameter.
  - Output: unique arbitrage-free price and replicating strategy for the option.
- But: finance $\neq$ physics; every model is a useful approximation at best. $\longrightarrow$ substantial **model uncertainty**.
- Challenges:
  - Consistently assess impact of model uncertainty?
  - Robust hedging strategies that take model uncertainty into account?
- Most widely used approach in the literature: Uncertain volatility model. [Avellaneda/Levy/Paras '95; Lyons '95]

Uncertain volatility model

- Volatility process $(\sigma_t)_{t\in[0,T]}$ evolves in band $[\sigma_{\min}, \sigma_{\max}]$; no assumptions on dynamics.
- Typical objective: find superhedging strategy for all possible volatilities. $\longrightarrow$ worst-case approach: infinite risk *and* uncertainty aversion.
- Yields robust no-arbitrage bounds.

Uncertain volatility model

- Volatility process $(\sigma_t)_{t\in[0,T]}$ evolves in band $[\sigma_{\min}, \sigma_{\max}]$; no assumptions on dynamics.
- Typical objective: find superhedging strategy for all possible volatilities. $\longrightarrow$ worst-case approach: infinite risk *and* uncertainty aversion.
- Yields robust no-arbitrage bounds.
- Drawbacks:
  - Ad hoc distinction between "possible" and "impossible" models.
  - All "possible" models are taken equally seriously.
  - No control over P&L if volatility leaves given band.

Uncertain volatility model

- Volatility process $(\sigma_t)_{t\in[0,T]}$ evolves in band $[\sigma_{\min}, \sigma_{\max}]$; no assumptions on dynamics.
- Typical objective: find superhedging strategy for all possible volatilities. $\longrightarrow$ worst-case approach: infinite risk *and* uncertainty aversion.
- Yields robust no-arbitrage bounds.
- Drawbacks:
  - Ad hoc distinction between “possible” and “impossible” models.
  - All “possible” models are taken equally seriously.
  - No control over P&L if volatility leaves given band.
- Interpolation between classical approach (one model, zero uncertainty) and worst-case approach?

Variational preferences

- Maccheroni/Marinacci/Rustichini '06: Decision-theoretic axioms suggest representation

$$\inf_P \left(E^P\left[U(Y)\right] + \alpha(P)\right)$$

- Utility function U describes **risk aversion** in a given model.
- Penalty functional α describes **uncertainty aversion.**

Variational preferences

- Maccheroni/Marinacci/Rustichini ’06: Decision-theoretic axioms suggest representation

$$\inf_P \left(E^P\left[U(Y)\right] + \alpha(P)\right)$$

- Utility function U describes **risk aversion** in a given model.
- Penalty functional α describes **uncertainty aversion**.
- Classical approach: infinite penalty for all but one model P.
- Worst-case approach: no penalty for a class $\mathfrak{P}$ of plausible models; infinite penalty otherwise.
- “Smooth” alternatives?

Moderate uncertainty aversion?

Multiplier preferences of Hansen/Sargent ’01:

- Models penalized by relative entropy with respect to a **reference model**.
- No strict line between “possible” and “impossible” models.
- Penalty is only finite for absolutely continuous measures. Not applicable to volatility uncertainty.

Moderate uncertainty aversion?

Multiplier preferences of Hansen/Sargent ’01:

- Models penalized by relative entropy with respect to a **reference model**.
- No strict line between “possible” and “impossible” models.
- Penalty is only finite for absolutely continuous measures. Not applicable to volatility uncertainty.

This paper:

- Choose similar penalty.

Hedging Problem

General approach

1. Choose a (big) class of plausible models.
2. Choose a reference model (“best guess”).
3. Choose a penalty functional, e.g., some “distance” to the reference model. Describes how seriously you take other models.
4. For each hedging strategy, compute its performance in the above sense (variational preferences).
5. Find an optimal strategy for this criterion.

Plausible models

- Dynamic trading in stock and call: model joint dynamics of stock S and implied volatility Σ of the call. [Lyons '97, Schönbucher '99, ...]
- (S, Σ): canonical process on $C([0,T];\mathbb{R}^2)$.
- $\mathfrak{P}$: probability measures P under which $S, \Sigma > 0$ and

$$\mathrm{d}S_t = S_t \sigma_t^P \,\mathrm{d}W_t^0,$$

$$\mathrm{d}\Sigma_t = \nu_t^P \,\mathrm{d}t + \eta_t^P \,\mathrm{d}W_t^0 + \sqrt{\xi_t^P} \,\mathrm{d}W_t^1,$$

for suitable σ^P, ν^P, η^P, and $\xi^P \geq 0$.

Plausible models

- Dynamic trading in stock and call: model joint dynamics of stock S and implied volatility Σ of the call. [Lyons '97, Schönbucher '99, ...]

- (S, Σ): canonical process on $C([0,T];\mathbb{R}^2)$.

- $\mathfrak{P}$: probability measures P under which $S, \Sigma > 0$ and

$$\mathrm{d}S_t = S_t \sigma_t^P \,\mathrm{d}W_t^0,$$
$$\mathrm{d}\Sigma_t = \nu_t^P \,\mathrm{d}t + \eta_t^P \,\mathrm{d}W_t^0 + \sqrt{\xi_t^P}\,\mathrm{d}W_t^1,$$

  for suitable σ^P, ν^P, η^P, and $\xi^P \geq 0$.

- Want call price $C_t = \mathcal{C}(t, S_t, \Sigma_t)$ to be drift-less like S.

- Itô on $\mathcal{C}(t, S_t, \Sigma_t)$ yields nonlinear *drift condition*:

$$\underbrace{\nu_t^P\,\mathcal{C}_\Sigma}_{\textbf{vega}} + \tfrac{1}{2}S_t^2\,\underbrace{\mathcal{C}_{SS}}_{\textbf{gamma}}((\sigma_t^P)^2 - \Sigma_t^2) + \sigma_t^P\eta_t^P S_t\,\underbrace{\mathcal{C}_{S\Sigma}}_{\textbf{vanna}} + \tfrac{1}{2}((\eta_t^P)^2 + \xi_t^P)\,\underbrace{\mathcal{C}_{\Sigma\Sigma}}_{\textbf{volga}} = 0.$$

Reference model

- Recalibrated Black–Scholes.
- Corresponds to $\nu^P = \eta^P = \xi^P = 0$ and $\sigma^P = \Sigma$.

"The future implied volatility stays at its currently observed level."

- Benchmark case; general approach extends to more realistic reference models.

P&L process

- Non-traded asset:
  - Payoff $h(S_T)$.
  - $\mathcal{V}(\cdot, \cdot, \sigma)$: smooth solution to the Black–Scholes PDE with volatility σ and terminal condition h.
  - Time-t reference value: $V_t = \mathcal{V}(t, S_t, \Sigma_t)$ ("mark-to-model").
  - To be hedged.
- P&L process:

$$Y_t^{\theta,\phi} = V_0 + \int_0^t \theta_u \, \mathrm{d}S_u + \int_0^t \phi_u \, \mathrm{d}C_u - V_t.$$

- Y_T is actual P&L at maturity because $V_T = \mathcal{V}(T, S_T, \Sigma_T) = h(S_T)$.

Penalty functional

- Penalize “mean-square” deviations from the reference BS model:

$$\alpha^{\psi}(P) = \frac{1}{2\psi} E^P \left[\int_0^T U'(Y_t^{\theta,\phi}) \left\{ (\nu_t^P)^2 + (\sigma_t^P - \Sigma_t)^2 + (\eta_t^P)^2 + (\xi_t^P)^2 \right\} \mathrm{d}t \right]$$

- Recall:

  ν^P: drift of implied volatility
  σ^P: spot volatility
  η^P: correlated volatility of implied volatility
  ξ^P: uncorrelated squared volatility of implied volatility

- $\psi > 0$ measures magnitude of uncertainty aversion.

Hedging problem

- Objective:

$$J^{\psi}(\theta,\phi,P)$$
$$= E^P\left[U(Y_T^{\theta,\phi}) + \frac{1}{2\psi}\int_0^T U'(Y_t^{\theta,\phi})\left\{(\nu_t^P)^2 + (\sigma_t^P - \Sigma_t)^2 + (\eta_t^P)^2 + (\xi_t^P)^2\right\}\,\mathrm{d}t\right]$$

- Hedging problem:

$$v(\psi) = \sup_{\theta,\phi}\inf_P J^{\psi}(\theta,\phi,P)$$

- To obtain explicit results: study limit $\psi \downarrow 0$.

Hedging problem

- Objective:

$$J^{\psi}(\theta, \phi, P)$$
$$= E^{P}\left[U(Y_T^{\theta,\phi}) + \frac{1}{2\psi}\int_0^T U'(Y_t^{\theta,\phi})\left\{(\nu_t^P)^2 + (\sigma_t^P - \Sigma_t)^2 + (\eta_t^P)^2 + (\xi_t^P)^2\right\} \mathrm{d}t\right]$$

- Hedging problem:

$$v(\psi) = \sup_{\theta,\phi} \inf_{P} J^{\psi}(\theta, \phi, P)$$

- To obtain explicit results: study limit $\psi \downarrow 0$.
- Goal:
  - Expansion for small uncertainty aversion:

$$v(\psi) = U(0) - \quad (?) \quad \psi + o(\psi).$$

  - Leading-order optimal strategy: $(\theta^{\psi}, \phi^{\psi})$ such that

$$v(\psi) = \inf_{P} J^{\psi}(\theta^{\psi}, \phi^{\psi}, P) + o(\psi).$$

Results

Almost optimality of delta-vega hedging

Under regularity and integrability assumptions…

- Dynamically recalibrated delta-vega hedge is optimal at the leading-order $O(\psi)$:
  - # of calls $\phi^\star = \mathcal{V}_\Sigma / \mathcal{C}_\Sigma$ neutralizes vega.
  - # of shares $\theta^\star = \mathcal{V}_S - \phi^\star \mathcal{C}_S$ neutralizes delta.
  - Greeks $\mathcal{V}_\Sigma$, $\mathcal{C}_\Sigma$, $\mathcal{V}_S$, and $\mathcal{C}_S$ computed in BS model and evaluated at time t with current stock price S_t and current implied volatility Σ_t.
  - $\inf_P J^\psi(\theta^\star, \phi^\star, P) = v(\psi) + o(\psi)$.

Almost optimality of delta-vega hedging

Under regularity and integrability assumptions...

- Dynamically recalibrated delta-vega hedge is optimal at the leading-order $O(\psi)$:
  - # of calls $\phi^\star = \mathcal{V}_\Sigma / \mathcal{C}_\Sigma$ neutralizes vega.
  - # of shares $\theta^\star = \mathcal{V}_S - \phi^\star \mathcal{C}_S$ neutralizes delta.
  - Greeks $\mathcal{V}_\Sigma$, $\mathcal{C}_\Sigma$, $\mathcal{V}_S$, and $\mathcal{C}_S$ computed in BS model and evaluated at time t with current stock price S_t and current implied volatility Σ_t.
  - $\inf_P J^\psi(\theta^\star, \phi^\star, P) = v(\psi) + o(\psi)$.
- Value has asymptotic expansion of the form

$$v(\psi) = U(0) - U'(0)\widetilde{w}\psi + o(\psi) \quad \text{as} \quad \psi \downarrow 0.$$

Almost optimality of delta-vega hedging

Under regularity and integrability assumptions...

- Dynamically recalibrated delta-vega hedge is optimal at the leading-order $O(\psi)$:
  - # of calls $\phi^\star = \mathcal{V}_\Sigma / \mathcal{C}_\Sigma$ neutralizes vega.
  - # of shares $\theta^\star = \mathcal{V}_S - \phi^\star \mathcal{C}_S$ neutralizes delta.
  - Greeks $\mathcal{V}_\Sigma$, $\mathcal{C}_\Sigma$, $\mathcal{V}_S$, and $\mathcal{C}_S$ computed in BS model and evaluated at time t with current stock price S_t and current implied volatility Σ_t.
  - $\inf_P J^\psi(\theta^\star, \phi^\star, P) = v(\psi) + o(\psi)$.
- Value has asymptotic expansion of the form

$$v(\psi) = U(0) - U'(0)\widetilde{w}\psi + o(\psi) \quad \text{as} \quad \psi \downarrow 0.$$

- What's $\widetilde{w}$? First look at "worst-case model".

"Worst-case model"

- Vega-gamma-vanna-volga vectors for non-traded option and call:

$$\mathbf{v} = \big(\mathcal{V}_\Sigma, \Sigma S^2 \mathcal{V}_{SS}, \Sigma S \mathcal{V}_{S\Sigma}, \frac{1}{2}\mathcal{V}_{\Sigma\Sigma}\big),$$
$$\mathbf{c} = \big(\mathcal{C}_\Sigma, \Sigma S^2 \mathcal{C}_{SS}, \Sigma S \mathcal{C}_{S\Sigma}, \frac{1}{2}\mathcal{C}_{\Sigma\Sigma}\big).$$

- Candidate feedback control for fictitious adversary:

$$(\nu^\psi, \sigma^\psi, \eta^\psi, \xi^\psi) = (0, \Sigma, 0, 0) + \widetilde{\boldsymbol{\zeta}}\psi,$$

where $\widetilde{\boldsymbol{\zeta}} = (\widetilde{\nu}, \widetilde{\sigma}, \widetilde{\eta}, \widetilde{\xi})$ is the solution to the linearly constrained quadratic minimization problem

$$\text{minimize} \quad \frac{1}{2}\left|\widetilde{\boldsymbol{\zeta}}\right|^2 - \mathbf{v}\cdot\widetilde{\boldsymbol{\zeta}} \quad \text{subject to} \quad \mathbf{c}\cdot\widetilde{\boldsymbol{\zeta}} = 0 \text{ and } \widetilde{\zeta}_4 \geq 0.$$

"Worst-case model"

- Vega-gamma-vanna-volga vectors for non-traded option and call:

$$\mathbf{v} = \big(\mathcal{V}_{\Sigma}, \Sigma S^2 \mathcal{V}_{SS}, \Sigma S \mathcal{V}_{S\Sigma}, \frac{1}{2}\mathcal{V}_{\Sigma\Sigma}\big),$$
$$\mathbf{c} = \big(\mathcal{C}_{\Sigma}, \Sigma S^2 \mathcal{C}_{SS}, \Sigma S \mathcal{C}_{S\Sigma}, \frac{1}{2}\mathcal{C}_{\Sigma\Sigma}\big).$$

- Candidate feedback control for fictitious adversary:

$$(\nu^{\psi}, \sigma^{\psi}, \eta^{\psi}, \xi^{\psi}) = (0, \Sigma, 0, 0) + \widetilde{\boldsymbol{\zeta}}\psi,$$

  where $\widetilde{\boldsymbol{\zeta}} = (\widetilde{\nu}, \widetilde{\sigma}, \widetilde{\eta}, \widetilde{\xi})$ is the solution to the linearly constrained quadratic minimization problem

$$\text{minimize} \quad \frac{1}{2}\left|\widetilde{\boldsymbol{\zeta}}\right|^2 - \mathbf{v}\cdot\widetilde{\boldsymbol{\zeta}} \quad \text{subject to} \quad \mathbf{c}\cdot\widetilde{\boldsymbol{\zeta}} = 0 \text{ and } \widetilde{\zeta}_4 \geq 0.$$

- But: $(\nu^{\psi}, \sigma^{\psi}, \eta^{\psi}, \xi^{\psi})$ does not satisfy the original nonlinear constraint *exactly*. Modify at order $O(\psi^2)$.

Value expansion

- Recall value expansion: $v(\psi) = U(0) - U'(0)\widetilde{w}\psi + o(\psi)$.
- $\widetilde{\sigma}$, $\widetilde{\eta}$, $\widetilde{\xi}$: deviations of "worst-case model" P^ψ from reference BS model:

spot vol	correlated vol of IV	uncorrelated vol^2 of IV
$\sigma^{P^\psi} \approx \Sigma + \widetilde{\sigma}\psi$	$\eta^{P^\psi} \approx \widetilde{\eta}\psi$	$\xi^{P^\psi} \approx \widetilde{\xi}\psi$

- $\widetilde{w}$ has probabilistic representation:

$$\widetilde{w} = \frac{1}{2}E\left[\int_0^T \widetilde{g}(t, S_t, \Sigma_0)\,\mathrm{d}t\right]$$

$$\begin{aligned}\widetilde{g}(t,S,\Sigma) = \Sigma S^2\left(\mathcal{V}_{SS} - \phi^\star\mathcal{C}_{SS}\right)\widetilde{\sigma} \quad & \textbf{–(net gamma)} \times \textbf{spot vol deviation}\\ +\Sigma S\left(\mathcal{V}_{S\Sigma} - \phi^\star\mathcal{C}_{S\Sigma}\right)\widetilde{\eta} \quad & \textbf{–(net vanna)} \times \textbf{correlated vol}\\ +\tfrac{1}{2}\left(\mathcal{V}_{\Sigma\Sigma} - \phi^\star\mathcal{C}_{\Sigma\Sigma}\right)\widetilde{\xi} \quad & \textbf{–(net volga)} \times \textbf{uncorrelated vol}^{\mathbf{2}}.\end{aligned}$$

- Independent of risk aversion due to complete reference model.

Indifference prices

- Indifference ask price for option V:

$$p_a(\psi) = \mathcal{V} + \widetilde{w}\psi + o(\psi)$$

- $\widetilde{w}\psi$ is "compensation" for model uncertainty.
  - Independent of the utility function, like the value V in the complete reference model.
- Sanity check:
  - $\widetilde{w} = 0$ if C is a call and V is a (multiple of a) put with matching strikes and maturities.
  - Model-free hedge by put-call parity.

Summary

Summary

- Preferences for moderate uncertainty aversion: models are penalized according to their distance from a reference model.
- Reference Black–Scholes model is dynamically recalibrated to the market price of the traded option.
- Impact of uncertainty depends on disparity between the vegas, gammas, vannas, and volgas of the non-traded and traded options.
- Delta-vega hedging arises naturally as leading-order optimal strategy.

Selected references

- M. Avellaneda, C. Friedman, R. Holmes, and D. Samperi.
  Calibrating volatility surfaces via relative-entropy minimization.
  Appl. Math. Finance, 4(1):37–64, 1997.
- M. Avellaneda, A. Levy, and A. Paras.
  Pricing and hedging derivative securities in markets with uncertain volatilities.
  Appl. Math. Finance, 2(2):73–88, 1995.
- M. Avellaneda and A. Paras.
  Managing the volatility risk of portfolios of derivative securities: the Lagrangian uncertain volatility model.
  Appl. Math. Finance, 3(1):21–52, 1996.
- R. Cont.
  Model uncertainty and its impact on the pricing of derivative instruments.
  Math. Finance, 16(3):519–547, 2006.
- J.-P. Fouque and B. Ren.
  Approximation for option prices under uncertain volatility.
  SIAM J. Financ. Math., 5(1):260–383, 2014.
- S. Herrmann and J. Muhle-Karbe.
  Model uncertainty, dynamic recalibration, and the emergence of delta-vega hedging.
  Finance Stoch., 2017+.
- S. Herrmann, J. Muhle-Karbe, and F. Seifried.
  Hedging with small uncertainty aversion.
  Finance Stoch., 21(1):1–64, 2017.
- N. El Karoui, M. Jeanblanc, and S. Shreve.
  Robustness of the Black and Scholes formula.
  Math. Finance, 8(2):93–126, 1998.
- T. Lyons.
  Uncertain volatility and the risk-free synthesis of derivatives.
  Appl. Math. Finance, 2(2):117–133, 1995.
- F. Maccheroni, M. Marinacci, and A. Rustichini.
  Ambiguity aversion, robustness, and the variational representation of preferences.
  Econometrica, 74(6):1447–1498, 2006.