Fluctuations of Rank Based Stochastic Differential Equations

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Overview

Rank Based Stochastic Differential Equations

Consider the following n interacting diffusions(particles) on the real line where the i^{th} diffusion evolves according to the SDE

$$\mathrm{d}X_i(t) = b\Big(F_n\big(t, X_i(t)\big)\Big)\,\mathrm{d}t + \sigma\Big(F_n\big(t, X_i(t)\big)\Big)\,\mathrm{d}W_i(t) \tag{1}$$

Here b, σ are lipschitz continuous functions from $[0,1]$ to $\mathbb{R}$, $F_n(t,x)$ is the empirical cdf $\frac{\sum_{i=1}^{n}\mathbb{I}\left(X_i(t)\leq x\right)}{n}$. We observe that the drift and volatility terms of the i^{th} particle depends on its rank. If Y_1, Y_2 and Y_3 are 3 real numbers with $Y_3 < Y_1 < Y_2$, then Rank(Y_1)=2, Rank(Y_2)=3 and Rank(Y_3)=1. This implies $b\Big(F_n\big(t, X_i(t)\big)\Big) = b\big(\frac{rank(X_i(t))}{n}\big)$ and we have a similar term for the volatility coefficient.

Law of Large numbers

Theorem (Shkolnikov '12, Jourdain, Reygner '13)

For the aforementioned interaction diffusions with b,σ *continuous and* $X_i(0), i = 1, 2, \ldots, n$ *IID drawn from* λ*, then* $F_n(t,x) \to F(t,x)$ *where* $F(t,x)$ *is a non random CDF and satisfies the following porous medium PDE with the initial condition* $F(0,x) = F_\lambda(x)$

$$F_t(t,x) = -\Big(b\big(F(t,x)\big)F_x(t,x)\Big) + \Big(\frac{\sigma^2\big(F(t,x)\big)}{2}F_x(t,x)\Big)_x \tag{2}$$

Derivation of PME

For any test function $f(x) \in C_c^\infty$, we have

$$\int_{\mathbb{R}} f(x)\mathrm{d}\big(F_n(t,x) - F_n(0,x)\big) = \frac{\sum_{i=1}^{n} \Big(f\big(X_i(t)\big) - f\big(X_i(0)\big)\Big)}{n}$$

$$= \int_0^t \int_{\mathbb{R}} \left(f'(x)b\big(F_n(s,x)\big) + f''(x)\frac{\sigma^2\big(F_n(s,x)\big)}{2}\right)\mathrm{d}F_n(s,x)\mathrm{d}s \qquad (3)$$

$$+ \sum_{i=1}^{n} \int_0^t \frac{f'\big(X_i(s)\big)\sigma\Big(F_n\big(s, X_i(s)\big)\Big)dW_i(s)}{n}$$

Now as $n \to \infty$, we expect $F_n(t,x)$ to converge to $F(t,x)$ and we also expect the integrals to converge appropriately.

Proof Cont'd

Upon letting $n \to \infty$, we obtain

$$\int_{\mathbb{R}} f(x)\mathrm{d}\big(F(t,x) - F(0,x)\big) = \\ = \int_0^t \int_{\mathbb{R}} \left(f'(x) b\big(F(s,x)\big) + f''(x) \frac{\sigma^2\big(F(s,x)\big)}{2} \right) \mathrm{d}F(s,x)\mathrm{d}s \tag{4}$$

Differentiating wrt t and using the fact that the limiting distribution $F(t,x)$ has a density $F_x(t,x)$, we get

$$\int_{\mathbb{R}} f(x) F_{xt}(t,x)\mathrm{d}x = \\ = \int_{\mathbb{R}} \left(f'(x) b\big(F(t,x)\big) + f''(x) \frac{\sigma^2\big(F(t,x)\big)}{2} \right) F_x(t,x)\mathrm{d}x \tag{5}$$

Integration by parts gives us the Porous Medium PDE.

Proof Cont'd

What happened to the Martingale Term ?

The Martingale term $M_t^n = \sum_{i=1}^{n} \int_0^t \frac{f'\left(X_i(s)\right)\sigma\left(F_n\left(s,X_i(s)\right)\right)dW_i(s)}{n}$ vanishes as $n \to \infty$. To see this, look at $\left\langle M^n \right\rangle_t$

$$\left\langle M^n \right\rangle_t = \int_0^t \int_{\mathbb{R}} \frac{\left(f'(x)\sigma(F_n(s,x))\right)^2}{n} dF_n(s,x)ds \to 0 \; as \; n \to \infty \qquad (6)$$

We have $F_n(t,x) \to F(t,x)$, the next natural question to ask is, What can we say about the fluctuations $G_n(t,x) = \sqrt{n}\left(F_n(t,x) - F(t,x)\right)$?

Theorem (Kolli, Shkolnikov '16)

$G_n(t,x) \to G(t,x)$, where G(t,x) is the mild solution of the SPDE with the initial condition $G(0,x) = \beta\big(F_\lambda(x)\big)$

$$G_t(t,x) = -\Big(b\big(F(t,x)\big)G(t,x)\Big)_x + \frac{\Big(\sigma^2\big(F(t,x)\big)G(t,x)\Big)_{xx}}{2} + \sigma\big(F(t,x)\big)\sqrt{F_x(t,x)}\,\dot{W}(t,x) \quad (7)$$

where $\dot{W}$ is space time white noise and β is an Independent standard Brownian Bridge.

Solution of the SPDE

,We can solve the SPDE explicitly and the solution is as follows

$$G(t,x) = \int_{\mathbb{R}} p(0,y,t,x) G(0,y) \mathrm{d}y + \int_0^t \int_{\mathbb{R}} \sigma\big(F(s,y)\big) \sqrt{F_x(s,y)} p(s,y,t,x) \mathrm{d}W(s,y) \tag{8}$$

where $p(s,y,t,x)$ is the transition density of the diffusion

$$\mathrm{d}\bar{X}_i(t) = b\big(F(t,\bar{X}_i(t))\big)\,\mathrm{d}t + \sigma\big(F(t,\bar{X}_i(t))\big)\,\mathrm{d}W_i(t) \tag{9}$$

Proof

There are two parts in the proof

- Tightness
- Identification of the limit

As $n \to \infty$, For every i, we expect $X_i(t)$ to fluctuate around $\bar{X}_i(t)$. The SDEs for $X_i(t)$ and $\bar{X}_i(t)$ are as follows

$$\mathrm{d}X_i(t) = b\big(F_n\big(t, X_i(t)\big)\big)\,\mathrm{d}t + \sigma\big(F_n\big(t, X_i(t)\big)\big)\,\mathrm{d}W_i(t) \tag{10}$$

$$\mathrm{d}\bar{X}_i(t) = b\big(F(t, \bar{X}_i(t))\big)\,\mathrm{d}t + \sigma\big(F(t, \bar{X}_i(t))\big)\,\mathrm{d}W_i(t) \tag{11}$$

We expect $X_i(t)$ to be close to $\bar{X}_i(t)$ and the following propogation of chaos estimate gives us the exact sense in which the two particles are close

Brief review of Wasserstein distance on the Real line

Let $F(x)$ and $G(x)$ be two Probability distributions on the Real line, then the p Wasserstein distance between F and G is defined as follows

$$W_p^p(F,G) = \int_0^1 \left|F^{-1}(t) - G^{-1}(t)\right|^p \mathrm{d}t \tag{12}$$

In view of Kantorovich duality, the 1 Wasserstein distance admits the following representation

$$W_1(F,G) = \int_{-\infty}^{\infty} \left|F(x) - G(x)\right| dx \tag{13}$$

Propogation of Chaos Estimate

Theorem (Kolli, Shkolnikov '16)

For all $p > 0$ and $T > 0$ there exists a constant $C = C(p, T) < \infty$ such that

$$\sum_{i=1}^{n} \mathbb{E}\left[\sup_{0 \le t \le T} \left| X_i(t) - \bar{X}_i(t) \right|^p \right] \le \frac{C}{n^{p/2-1}}. \tag{14}$$

In particular, when $p \ge 1$ one has

$$\mathbb{E}\left[\sup_{0 \le t \le T} W_p^p\big(F_n(t,x), \overline{F_n}(t,x)\big) \right] \le C\, n^{-p/2}. \tag{15}$$

where $F_n(t,x)$ and $\overline{F_n}(t,x)$ are the empirical CDFs corresponding to the particle systems $X_i(t)$ and $\bar{X}_i(t)$ respectively.

We remark that tightness is a simple consequence of the above estimate.

Identification of the limit

for any smooth test function h(t,x)

$$\int_{\mathbb{R}} h(t,x)\big(G_n(t,x) - G_n(0,x)\big)dx = \int_0^t \int_{\mathbb{R}} h_t(s,x)G_n(s,x)dxds$$

$$+ \int_0^t \int_{\mathbb{R}} \sqrt{n}\Big(h_x(s,x)(B_n - B)\big(F_n(s,x)\big) + h_{xx}(s,x)(\Sigma_n - \Sigma)\big(F_n(s,x)\big)\Big)dxds$$

$$+ \int_0^t \int_{\mathbb{R}} \sqrt{n}h_x(s,x)\Big(B\big(F_n(s,x)\big) - B\big(F(s,x)\big)\Big)dxds$$

$$+ \int_0^t \int_{\mathbb{R}} \sqrt{n}h_{xx}(s,x)\Big(\Sigma\big(F_n(s,x)\big) - \Sigma\big(F(s,x)\big)\Big)dxds + Martingale\ term(M_t^n) \tag{16}$$

Proof Cont'd

$$Martingale\ term(M_t^n) = -\sum_{i=1}^{n} \int_0^t \frac{h(s, X_i(s))\sigma(F_n(s, X_i(s)))dW_i(s)}{\sqrt{n}} \quad (17)$$

where $B(x) = \int_0^x b(y)dy$, $\Sigma(x) = \int_0^x \frac{\sigma^2(y)}{2}dy$, $B_n(\frac{i}{n}) = \frac{\sum_{j=1}^{i} b(\frac{j}{n})}{n}$ and $\Sigma_n(\frac{i}{n}) = \frac{\sum_{j=1}^{i} \sigma^2(\frac{j}{n})}{2n}$, We also notice that $(B_n - B)(F_n(s, x)) = O(\frac{1}{n})$ and similarly $(\Sigma_n - \Sigma)(F_n(s, x)) = O(\frac{1}{n})$. Upon letting $n \to \infty$, we expect $G_n(t, x)$ to converge to $G(t, x)$ and we also expect the integrals to converge appropriately.

Proof Cont'd

The limit G(t,x) satisfies the following

$$\int_{\mathbb{R}} h(t,x)\big(G(t,x) - G(0,x)\big)dx = \int_0^t \int_{\mathbb{R}} h_t(s,x)G(s,x)dxds$$
$$+ \int_0^t \int_{\mathbb{R}} \left(h_x(s,x)b\big(F(s,x)\big) + h_{xx}\frac{\sigma^2\big(F(s,x)\big)}{2}\right)G(s,x)dxds \tag{18}$$
$$+ \textit{Martingale term}(M_t)$$

Proof Cont'd

To find out the limit of the Martingale M_t^n, we look at the quadratic variation $\left\langle M^n \right\rangle_t$

$$\left\langle M^n \right\rangle_t = \int_0^t \int_{\mathbb{R}} \big(h(s,x)\sigma(F_n(s,x))\big)^2 dF_n(s,x)ds \tag{19}$$

Now as $n \to \infty$, $\left\langle M^n \right\rangle_t \to \left\langle M \right\rangle_t$ and $\left\langle M \right\rangle_t$ is as follows

$$\left\langle M \right\rangle_t = \int_0^t \int_{\mathbb{R}} \big(h(s,x)\sigma(F(s,x))\big)^2 F_x(s,x)dxds \tag{20}$$

From this we infer that

$$M_t = \int_0^t \int_{\mathbb{R}} h(s,x)\sigma\big(F(t,x)\big)\sqrt{F_x(t,x)}dW(s,x) \tag{21}$$

Proof Cont'd

Differentiating wrt t and Integration by parts gives us the SPDE that we want

$$G_t(t,x) = -\Big(b\big(F(t,x)\big)G(t,x)\Big)_x + \frac{\Big(\sigma^2\big(F(t,x)\big)G(t,x)\Big)_{xx}}{2} + \sigma\big(F(t,x)\big)\sqrt{F_x(t,x)}\dot{W}(t,x) \tag{22}$$

References

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Thank You