A General Valuation Framework for SABR and Stochastic Local Volatility Models

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Mathematical Finance, Probability, and Partial Differential Equations Conference
Rutgers, The State University of New Jersey

May 19th, 2017

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Overview

- Stochastic local volatility (SLV) models, including:
  - SABR (variants: shifted, lambda, Heston, etc.)
  - Quadratic SLV (Lipton), root quadratic SLV
- Includes pure stochastic volatility (SV) as special case:
  - Heston, Jacobi, Hull-White, 3/2, Stein-Stein, 4/2, Scott, α-Hypergeometric, etc.
- Includes mean-reverting Commodity models:
  - Ornstein-Uhlenbeck with SV, mean-reverting SABR, etc.
- Contracts: European, Bermudan/American, Barrier, Asian, Parisian/Occupation time, Lookback

Stochastic Local Volatility

- Local volatility (Dupire (1994)[4], Derman et al (1996)[5]), for "perfect calibration":

$$LV: \quad dS_t = S_t \mu dt + \sigma_{LV}(S_t, t) S_t dW_t$$

- Stochastic volatility for realistic surface dynamics:

$$SV: \quad \begin{cases} dS_t = S_t \mu dt + m(v_t) S_t dW_t^{(1)}, \\ dv_t = \mu(v_t) dt + \sigma(v_t) dW_t^{(2)} \end{cases}$$

- Stochastic local volatility to unite the two:

$$SLV: \quad \begin{cases} dS_t = \omega(S_t, v_t) dt + m(v_t) \Gamma(S_t) dW_t^{(1)}, \\ dv_t = \mu(v_t) dt + \sigma(v_t) dW_t^{(2)} \end{cases}$$

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[4] Dupire, B. (1994). Pricing with a Smile. Risk Magazine.

[5] Derman, E., Kani, I., and Chriss, N. (1996). Implied trinomial tress of the volatility smile. The Journal of Derivatives, 3(4), 7-22.

Applicable SLV Dynamics

SABR (Hagan et al. (2002))	$dS_t = v_t S_t^{\beta} dW_t^{(1)}$ $dv_t = \alpha v_t dW_t^{(2)}$	$\beta \in [0,1)$ $\alpha, v_0 > 0$
$\lambda-$SABR (Henry-Labodere (2005))	$dS_t = v_t S_t^{\beta} dW_t^{(1)}$ $dv_t = \lambda(\theta - v_t)dt + \alpha v_t dW_t^{(2)}$	$\beta \in [0,1)$ $\lambda, \theta, \alpha, v_0 > 0$
Shifted SABR (Antonov et al. (2015))	$dS_t = v_t (S_t + s)^{\beta} dW_t^{(1)}$ $dv_t = \alpha v_t dW_t^{(2)}$	$\beta \in [0,1)$ $s, \alpha, v_0 > 0$
Heston-SABR (Van Der Stoep et al. (2014))	$dS_t = rS_t dt + \sqrt{v_t} S_t^{\beta} dW_t^{(1)}$ $dv_t = \eta(\theta - v_t)dt + \alpha\sqrt{v_t} dW_t^{(2)}$	$r \in \mathbb{R}, \beta \in [0,1)$ $\eta, \theta, \alpha, v_0 > 0$
Quadratic SLV (Lipton (2002))	$dS_t = rS_t dt + \sqrt{v_t}(aS_t^2 + bS_t + c) dW_t^{(1)}$ $dv_t = \eta(\theta - v_t)dt + \alpha\sqrt{v_t} dW_t^{(2)}$	$r \in \mathbb{R}, \beta \in [0,1)$ $a, \eta, \theta, \alpha, v_0 > 0,\ 4ac > b^2$
Exponential SLV	$dS_t = rS_t dt + m(v_t)(v_L + \theta \exp(-\lambda S_t)) dW_t^{(1)}$ $dv_t = \mu(v_t)dt + \sigma(v_t) dW_t^{(2)}$	$r \in \mathbb{R}, \lambda, v_L \geq 0$ $v_L + \theta \geq 0$
Root-Quadratic SLV	$dS_t = rS_t dt + m(v_t)\sqrt{aS_t^2 + bS_t + c}\ dW_t^{(1)}$ $dv_t = \mu(v_t)dt + \sigma(v_t) dW_t^{(2)}$	$r \in \mathbb{R}$ $a > 0, c \geq 0$
Tan-Hyp SLV	$dS_t = rS_t dt + m(v_t)\tanh(\beta S_t) dW_t^{(1)}$ $dv_t = \mu(v_t)dt + \sigma(v_t) dW_t^{(2)}$	$r \in \mathbb{R}$ $\beta \geq 0$
Mean-reverting-SABR	$dS_t = \kappa(\zeta - S_t)dt + m(v_t) S_t^{\beta} dW_t^{(1)}$ $dv_t = \mu(v_t)dt + \sigma(v_t) dW_t^{(2)}$	$r \in \mathbb{R}, \beta \in [0,1)$ $\kappa, \zeta, v_0 > 0$
4/2-SABR	$dS_t = rS_t dt + S_t^{\beta}[a\sqrt{v_t} + b/\sqrt{v_t}] dW_t^{(1)}$ $dv_t = \eta(\theta - v_t)dt + \alpha\sqrt{v_t} dW_t^{(2)}$	$r \in \mathbb{R}, \beta \in [0,1)$ $a, b, \eta, \theta, \alpha, v_0 > 0$

Table: Some stochastic local volatility models

Applicable SV Dynamics

Heston (Heston 1993)	$m(v)=\sqrt{v}, \quad dv_t=\eta(\theta-v_t)dt+\sigma_v\sqrt{v_t}dW_t^{(2)}$ $f(v)=v/\sigma_v, \quad h(v)=\eta(\theta-v)/\sigma_v$
4/2 (Grasselli 2016)	$m(v)=a\sqrt{v}+b/\sqrt{v}, \quad dv_t=\eta(\theta-v_t)dt+\sigma_v\sqrt{v_t}dW_t^{(2)}$ $f(v)=\frac{(av+b\log(v))}{\sigma_v}, \quad h(v)=\frac{\eta(a\theta-b)}{\sigma_v}-\frac{a\eta v}{\sigma_v}+(\frac{\eta\theta b}{\sigma_v}-\frac{b\sigma_v}{2})\frac{1}{v}$
Stein-Stein (Stein-Stein 1991)	$m(v)=v, \quad dv_t=\eta(\theta-v_t)dt+\sigma_v dW_t^{(2)}$ $f(v)=\frac{v^2}{2\sigma_v}, \quad h(v)=\frac{\sigma_v}{2}+\frac{\eta\theta v}{\sigma_v}-\frac{\eta v^2}{\sigma_v}$
3/2 (Lewis 2000)	$m(v)=1/\sqrt{v}, \quad d\widehat{v}_t=\widehat{\eta}[\widehat{\theta}-\widehat{v}_t]dt+\widehat{\sigma}_v\sqrt{\widehat{v}_t}dW_t^{(2)}$ $f(v)=\frac{\log(v)}{\widehat{\sigma}_v}, \quad h(v)=\left(\frac{\widehat{\eta}\widehat{\theta}}{\widehat{\sigma}_v}-\frac{\widehat{\sigma}_v}{2}\right)\frac{1}{v}-\frac{\widehat{\eta}}{\widehat{\sigma}_v}$
Hull-White (Hull-White 1987)	$m(v)=\sqrt{v}, \quad dv_t=a_v v_t dt+\sigma_v v_t dW_t^{(2)}$ $f(v)=\frac{2\sqrt{v}}{\sigma_v}, \quad h(v)=\left(\frac{a_v}{\sigma_v}-\frac{\sigma_v}{4}\right)\sqrt{v}$
Scott (Scott 1987)	$m(v)=\exp(v), \quad dv_t=\eta(\theta-v_t)dt+\sigma_v dW_t^{(2)}$ $f(v)=\frac{e^v}{\sigma_v}, \quad h(v)=e^v\left(\frac{\eta\theta}{\sigma_v}+\frac{\sigma_v}{2}-\frac{\eta v}{\sigma_v}\right)$
α-Hyper (Da Fonseca & Martini 2016)	$m(v)=\exp(v), \quad dv_t=(\eta-\theta\exp(a_v v_t))dt+\sigma_v dW_t^{(2)}$ $f(v)=\frac{e^v}{\sigma_v}, \quad h(v)=e^v\left(\frac{\eta}{\sigma_v}+\frac{\sigma_v}{2}\right)-\frac{\theta}{\sigma_v}e^{(a_v+1)v}$
Jacobi (Ackerer et al. 2016)	$dv_t=\kappa(\theta-v_t)dt+\alpha\sqrt{Q(v_t)}dW_t^{(2)}$ (see paper)

Table: Dynamics and variance transforms for some stochastic volatility models.

Technical Assumptions

- **Assumption 1.** Let $P_t\Phi(S,v) = \mathbb{E}[\Phi(S_t,v_t)|S_0 = S, v_0 = v]$, and for any $\Phi \in C_0([0,\infty)\times[0,\infty))$, we assume that (S_t,v_t) is a Feller process, i.e.
  - $P_t\Phi \in C_0([0,\infty)\times[0,\infty))$ for any $t \geq 0$
  - $\lim_{t\to 0} P_t\Phi(S,v) = \Phi(S,v)$ for any $(S,v) \in [0,\infty)\times[0,\infty)$.
- The Feller property guarantees that there exists a version of the process (S_t,v_t) with cádlág paths satisfying the strong Markov property.
- The family of $P_t\Phi(S,v)$ is determined by its infinitestimal generator $\mathcal{L}^S$:

$$\mathcal{L}^S\Phi(S,v) = \lim_{t\to 0+} \frac{(P_t\Phi - \Phi)(S,v)}{t}. \tag{1}$$

- **Assumption 2.** We assume that

$$\lim_{(S,v)\to(0,0)} \mathcal{L}^S\Phi(S,v) = 0. \tag{2}$$

SLV Methodology Outline

$$SLV: \quad \begin{cases} dS_t = \omega(S_t, v_t)dt + m(v_t)\Gamma(S_t)dW_t^{(1)}, \\ dv_t = \mu(v_t)dt + \sigma(v_t)dW_t^{(2)} \end{cases}$$

1. Transformation: $(S_t, v_t) \to (X_t, v_t)$, to decouple the local and stochastic volatility term $m(v_t)\Gamma(S_t)$
2. First layer approximation: $(\tilde{X}_t, v_{\alpha(t)})$, where $v_{\alpha(t)}$ is a locally consistent CTMC approximation to v_t.
3. Second layer approximation: $(\tilde{X}_t^{(N)}, v_{\alpha(t)})$, a nonlinear Regime Switching CTMC
4. Dimension Reduction: $(\tilde{X}_t^{(N)}, v_{\alpha(t)}) \to Y_t$, a one-dimensional CTMC
5. Pricing in the space of Y_t

Transformed Process

- First apply standard Cholesky decomposition

$$\begin{aligned} dS_t &= \omega(S_t, v_t)dt + m(v_t)\Gamma(S_t)dW_t^{(1)} \\ &= \omega(S_t, v_t)dt + m(v_t)\Gamma(S_t)(\rho dW_t^{(2)} + \sqrt{1-\rho^2}dW_t^{(*)}), \end{aligned}$$

  which contains two (independent) Brownian motions
- Goal is to obtain an auxiliary process of the form:

$$dF(S_t, v_t) = F^1(S_t, v_t)dt + F^2(S_t, v_t)dW_t^{(*)},$$

  driven by a single Brownian motion
- Such a process is amenable to a regime switching approximation

Transformed Process

Recall: $$dS_t = \omega(S_t, v_t)dt + m(v_t)\Gamma(S_t)dW_t^{(1)}$$

Lemma

Define the functions $g(x) := \int_\cdot^x \frac{1}{\Gamma(u)}du$ *and* $f(x) := \int_\cdot^x \frac{m(u)}{\sigma(u)}du$*. Then we have*

$$\begin{cases} d(g(S_t) - \rho f(v_t)) = \left(\frac{\omega(S_t, v_t)}{\Gamma(S_t)} - \frac{\Gamma'(S_t)}{2}m^2(v_t) - \rho h(v_t)\right)dt \\ \qquad\qquad\qquad\qquad\qquad\qquad + \sqrt{1-\rho^2}m(v_t)dW_t^*, \\ dv_t = \mu(v_t)dt + \sigma(v_t)dW_t^{(2)}, \end{cases}$$

where W_t^* *and* $W_t^{(2)}$ *are two independent Brownian motions, and*

$$h(v_t) = \mu(v_t)\frac{m(v_t)}{\sigma(v_t)} + \frac{1}{2}\big(\sigma(v_t)m'(v_t) - m(v_t)\big).$$

This defines the auxiliary process $X_t := g(S_t) - \rho f(v_t)$*.*

Markov Chain Review

- Consider a CTMC, $\alpha(t) \in \mathcal{M} := \{1, 2, \dots, m_0\}$
- Transition density at time t of $\alpha(t + \Delta t)|\alpha(s)$, $0 \le s \le t$, depends only on $\alpha(t)$
- Dynamics of $\alpha(t)$ captured by *rate matrix* $\mathbf{\Lambda} = [\lambda_{ij}]_{m_0 \times m_0}$
- λ_{ij} is transition rate from state i to j, and $\lambda_{ii} = -\sum_{j \neq i} \lambda_{ij}$
- In particular, $\forall i \neq j$:

$$\mathbb{Q}(\alpha(t + \Delta t) = j | \alpha(t) = i, \alpha(t'), 0 \le t' \le t) = \lambda_{ij} \Delta t + o(\Delta t).$$

- Closed-form probabilities in terms of matrix exponential:

$$\mathbf{P}(\Delta t) = \exp(\mathbf{\Lambda} \cdot \Delta t) := \sum_{k=0}^{\infty} \frac{(\mathbf{\Lambda} \cdot \Delta t)^k}{k!}$$

First Layer Approximation: Variance CTMC Approximation

- Define $v_{\alpha(t)}$, a CTMC approximating v_t, by defining a rate matrix $\mathbf{\Lambda} = [\lambda_{ij}]_{m_0 \times m_0}$ for $\alpha(t)$
- Local consistency: choose $\mathbf{\Lambda}$ so that first two moments of dv_t and $dv_{\alpha(t)}$ match locally
- For example: uniform grid $v_i = v_1 + \delta_v(i-1)$, $i = 1, \dots, m_0$

$$\lambda_{ij} = \begin{cases} \frac{-\mu(v_i)}{2\delta_v} + \frac{\sigma^2(v_i)}{2\delta_v^2}, & j = i-1, \\ -\frac{\sigma^2(v_i)}{\delta_v^2}, & j = i, \\ \frac{\mu(v_i)}{2\delta_v} + \frac{\sigma^2(v_i)}{2\delta_v^2}, & j = i+1 \end{cases}$$

- In practice, we apply the generator of Lo and Skindilias (2014), as in Kirkby et. al (2016) for pure SV, combined with nonuniform grid of Tavella and Randall (2000)

Second Layer Approximation: Regime Switching CTMC

- In first stage, we approximated v_t with $v_{\alpha(t)}$
- Process $(\tilde{X}_t, v_{\alpha(t)})$ defines a nonlinear regime switching (RS) model, which maps to $(\tilde{S}_t, v_{\alpha(t)})$
- Mapping is defined by

$$\tilde{X}_t := g(\tilde{S}_t) - \rho f(v_{\alpha(t)})$$

$$\tilde{S}_t = g^{-1}(\tilde{X}_t + \rho f(v_{\alpha(t)}))$$

- In particular, we have a new RS model for $\tilde{S}_t$,

$$d\tilde{S}_t = \omega(\tilde{S}_t, v_{\alpha(t)})dt + m(v_{\alpha(t)})\Gamma(\tilde{S}_t)dW_t^{(1)}, \qquad \tilde{S}_0 = S_0$$

Second Layer Approximation: Regime Switching CTMC

- We focus on the dynamics of $\tilde{X}_t$, with solution:

$$\tilde{X}_t = g(S_0) - \rho f(v_0) + \int_0^t \theta(\tilde{X}_t, v_{\alpha(t)})dt + \sqrt{1-\rho^2}\int_0^t m(v_{\alpha(t)})dW_t^*$$

- For each fixed state $v_{\alpha(t)}$, this process is a (time-homogeneous) diffusion
- Hence, we apply a second layer CTMC, with generators $\boldsymbol{G}^l = [q_{ij}^l]_{N\times N}$, for $l = 1, \dots, m_0$:

$$q_{kj}^l = \begin{cases} \frac{-\theta(x_k, v_l)}{2\delta_x} + \frac{\tilde{\sigma}^2(v_i)}{2\delta_x^2}, & j = k-1, \\ -\frac{\tilde{\sigma}^2(v_i)}{\delta_x^2}, & j = k, \\ \frac{\theta(x_k, v_l)}{2\delta_x} + \frac{\tilde{\sigma}^2(v_i)}{2\delta_x^2}, & j = k+1 \end{cases}$$

where $\tilde{\sigma}(v_l) := \sqrt{1-\rho^2}m(v_l)$

- New process: $(\tilde{X}^{(N)}, v_{\alpha(t)})$ is a RS-CTMC

Relation to Infinitesimal Generator

- The infinitesimal generator of $(\tilde{X}^{(N)}, v_{\alpha(t)})$:

$$
\begin{aligned}
\mathcal{L}_N^{m_0} V(x_k, v_l) &= \sum_{j=1}^{N} g_{kj}^l V_j^l + \sum_{n=1}^{m_0} \lambda_{ln} V_k^n \\
&= \left(g_{k,k-1}^l V_{k-1}^l + g_{kk}^l V_k^l + g_{k,k+1}^l V_{k+1}^l \right) \\
&\quad + \underbrace{\left(\lambda_{l,l-1} V_k^{l-1} + \lambda_{ll} V_k^l + \lambda_{l,l+1} V_k^{l+1} \right)} .
\end{aligned}
$$

- Generator of (X_t, v_t) given by

$$
\begin{aligned}
\mathcal{L} V(x, v) = \frac{1}{2}(1-\rho^2)[m(v)]^2 \frac{\partial^2 V}{\partial x^2} + \theta(x, v) \frac{\partial V}{\partial x} \\
+ \underbrace{\frac{1}{2}\sigma^2(v) \frac{\partial^2 V}{\partial v^2} + \mu(v) \frac{\partial V}{\partial v}} .
\end{aligned}
$$

Relation to Infinitesimal Generator

- Comparing, for example, the underlined terms:

$$\lambda_{l,l-1} V_k^{l-1} + \lambda_{ll} V_k^{l} + \lambda_{l,l+1} V_k^{l+1}$$
$$= \frac{\sigma^2(v_l)}{2} \left[\frac{V_k^{l-1} - 2V_k^{l} + V_k^{l-1}}{\delta_v^2} \right] + \mu(v_l) \left[\frac{V_k^{l+1} - V_k^{l-1}}{2\delta_v} \right]$$
$$\to \frac{1}{2}\sigma^2(v_l)\frac{\partial^2 V}{\partial v^2} + \mu(v_l)\frac{\partial V}{\partial v}, \quad \text{as } \delta_v \downarrow 0$$

- Analogous result for other terms as $\delta_x \downarrow 0$
- Hence weak convergence: $(\tilde{X}^{(N)}, v_{\alpha(t)}) \Longrightarrow (X_t, v_t)$, from which expected values (prices) converge to true values

Weak Convergence

Proposition

(Proposition 3) For each fixed m_0 *and* N*, let* $[x_1, x_N]$ *and* $[v_1, v_{m_0}]$ *be the truncation domains*[6] *for the processes* $\tilde{X}_t^{(N)}$ *and* $v_{\alpha(t)}^{m_0} \equiv v_{\alpha(t)}$ *respectively. For each* m_0 *and* N*, choose* $\mathbf{\Lambda} = (\lambda_{ij})_{m_0 \times m_0}$ *and* $\mathbf{G}_l = (q_{ij}^l)_{N \times N}$ *for* $l = 1, \ldots, m_0$ *as in the paper. Then*

$$(\tilde{X}_t^{(N)}, v_{\alpha(t)}^{m_0}) \Longrightarrow (X_t, v_t), \quad \text{as} \quad m_0, N \to \infty.$$

Here "$\Longrightarrow$*" indicates weak convergence of the regime-switching CTMC approximation to the true process.*

[6] We have the freedom to choose x_1, x_N, v_1, v_{m_0}, and we will choose them so that the domain of (X_t, v_t) is covered sufficiently.

Convergence Order

- Li and Zhang (2016)[7] consider an error estimate of the option value using the Markov chain approximation of Mijatovic and Pistorius (2013)[8].
- Using the spectral analysis, they show that for call/put-type payoffs, the convergence is of second order, while for digital-type payoffs, the convergence is only of first order in general.
- It is expected that the same convergence type would hold for the model considered in this paper. It would be very interesting to carry this out. We leave this as an interesting project for future studies.

[7] Li, L. and Zhang, G., 2016. Option Pricing in Some Non-Levy Jump Models. SIAM Journal on Scientific Computing, 38(4), pp.B539-B569.

[8] Mijatovic, A. and Pistorius, M., 2013. Continuously monitored barrier options under Markov processes. Mathematical Finance, 23(1), pp.1-38.

Transform to One-dimensional CTMC

- Process $(\tilde{X}^{(N)}, v_{\alpha(t)})$ defines a RS-CTMC
- Next apply theorem of Song et al (2016)[9] to convert to a 1-D CTMC, $\{Y_t, t \geq 0\}$, on $\mathcal{S}_Y := \{1, 2, ..., N \cdot m_0\}$
- First find a bijection between state space $\mathcal{S}_X \times \mathcal{M}$ of $(\tilde{X}^{(N)}, v_{\alpha(t)})$ to $\mathcal{S}_Y$ of Y_t
- Define the mapping $\phi : \mathcal{S}_X \times \mathcal{M} \to \mathcal{S}_Y$ by

$$\phi(x_k, l) = (l-1)N + k, \quad 1 \leq l \leq m_0, \; 1 \leq k \leq N$$

- Inverse $\phi^{-1} : \mathcal{S}_Y \to \mathcal{S}_X \times \mathcal{M}$ by

$$\phi^{-1}(n) = (x_k, l), \quad \text{for } n \in \mathcal{S}_Y$$

where k is the unique integer satisfying $n = (l-1)N + k$ for some $l \in \{1, 2, \ldots, m_0\}$.

[9] Song, Y., Cai, N. and Kou, S., 2016. A Unified Framework for Options Pricing under Regime Switching Models.

Transform to One-dimensional CTMC

Theorem

(Song et al (2016)) Define the $N \cdot m_0 \times N \cdot m_0$ rate matrix

$$\mathbf{G} = \begin{pmatrix} \lambda_{11}\mathbf{I}_N + \mathbf{G}_1 & \lambda_{12}\mathbf{I}_N & \cdots & \lambda_{1m_0}\mathbf{I}_N \\ \lambda_{21}\mathbf{I}_N & \lambda_{22}\mathbf{I}_N + \mathbf{G}_2 & \cdots & \lambda_{2m_0}\mathbf{I}_N \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{m_0 1}\mathbf{I}_N & \lambda_{m_0 2}\mathbf{I}_N & \cdots & \lambda_{m_0 m_0}\mathbf{I}_N + \mathbf{G}_{m_0} \end{pmatrix},$$

where $\mathbf{I}_N$ is the $N \times N$ identity matrix, $\mathbf{G}_l = (q^l_{kj})_{N\times N}$, and $\mathbf{\Lambda} = (\lambda_{k,j})_{m_0 \times m_0}$. Then we have

$$\mathbb{E}\left[\Psi(\tilde{X}^{(N)}, \alpha)|\alpha(0) = i, \tilde{X}_0^{(N)} = x_k\right]$$
$$= \mathbb{E}[\Psi \circ \phi^{-1}(Y)|Y_0 = (i-1)N + k],$$

for any path-dependent payoff function Ψ. Here we have defined $\tilde{X}^{(N)} := (\tilde{X}_t^{(N)})_{0\le t\le T}$, $\alpha := (\alpha(t))_{0\le t\le T}$, and $Y := (Y_t)_{0\le t\le T}$.

European Options

- Vanilla option prices for the underlying S_T can now be approximated with respect to

$$\tilde{S}_T^{(N)} := g^{-1}(\tilde{X}_T^{(N)} + \rho f(v_{\alpha(T)})). \tag{3}$$

- For example

$$\begin{aligned}
&\mathbb{E}\left[e^{-rT}\left(S_T - K\right)^+ \middle| v_0, S_0\right] \\
&\approx \mathbb{E}\left[e^{-rT}\left(\tilde{S}_T^{(N)} - K\right)^+ \middle| \alpha(0) = i, \tilde{X}_0^{(N)} = x_k\right] \\
&= e^{-rT} \cdot \mathbf{e}_{i,x_k} \cdot \exp(\mathbf{G}T) \cdot \mathbf{H}^{(1)}
\end{aligned}$$

where $\mathbf{e}_{i,x_k}$ is a $1 \times Nm_0$ vector with all entries equal to 0 except that the $(i-1)N + k$ entry is equal to 1, and $\mathbf{H}^{(1)}$ is an $Nm_0 \times 1$ vector with

$$H^{(1)}_{(l-1)N+j} = \begin{cases} \left(g^{-1}(\tilde{x}_j + \rho f(v_l)) - K\right)^+ & \text{for a call,} \\ \left(K - g^{-1}(\tilde{x}_j + \rho f(v_l))\right)^+ & \text{for a put.} \end{cases}$$

- By the continuity of $g^{-1}(\cdot)$, it follows from Proposition 3 and the continuous mapping theorem that

$$\tilde{S}_T^{(N)} \Longrightarrow S_T, \quad \text{as} \quad m_0, N \to \infty,$$

- It then follows that $\mathbb{E}[H(\tilde{S}_T^{(N)})] \to \mathbb{E}[H(S_T)]$ for any bounded payoff $H(\cdot)$ which is continuous on $C \subset [0, \infty)$ such that $\mathbb{Q}[S_T \in C] = 1$.

- Assuming that $\mathbb{E}[H(S_T)] < \infty$, for any $\epsilon > 0$ we can choose $\theta > 0$ such that $\mathbb{E}[H(S_T) - H(S_T)\mathbb{1}_{\{S_T \leq \theta\}}] < \epsilon$, and

$$\mathbb{E}\left[H(\tilde{S}_T^{(N)})\mathbb{1}_{\{S_T^{(N)} \leq \theta\}}\right] \to \mathbb{E}\left[H(S_T)\mathbb{1}_{\{S_T \leq \theta\}}\right].$$

- Hence, the Markov chain value approximation will converge for any finitely valued European option.

American and Barrier Options

- Define transition density $\mathbf{P}_Y(\Delta) := \exp(\mathbf{G}\Delta)$ of Y_Δ.
- Bermudan value recursion for $\mathcal{V}^{Ber}(\tilde{X}_0^{(N)}, \alpha_0, K) = \boldsymbol{\mathcal{V}}_0(n)$:

$$\begin{cases} \boldsymbol{\mathcal{V}}_M = \mathbf{H}^{(\mathbf{1})} \\ \boldsymbol{\mathcal{V}}_m = \max\{e^{-r\Delta}\mathbf{P}_Y(\Delta)\boldsymbol{\mathcal{V}}_{m+1},\ \mathbf{H}^{(\mathbf{1})}\}, & m = M-1, \dots, 0 \end{cases}$$

- Barrier option with knock-out barrier $B \geq 0$: define the indicator vector $\mathbf{1}_B$, where for each $n = (l-1)N + k \in \mathcal{S}_Y$,

$$\mathbf{1}_B(n) = \begin{cases} \mathbb{1}\{g(B) \leq x_k + \rho f(v_l)\}, & \text{down-and-out} \\ \mathbb{1}\{g(B) \geq x_k + \rho f(v_l)\}, & \text{up-and-out,} \end{cases}$$

- Recursion to find $\mathcal{V}^{Bar}(\tilde{X}_0^{(N)}, \alpha_0, B, K) = \boldsymbol{\mathcal{V}}_0(n)$

$$\begin{cases} \boldsymbol{\mathcal{V}}_M = \mathbf{H}^{(\mathbf{1})} \circ \mathbf{1}_B \\ \boldsymbol{\mathcal{V}}_m = e^{-r\Delta}\mathbf{P}_Y(\Delta)\boldsymbol{\mathcal{V}}_{m+1} \circ \mathbf{1}_B, & m = M-1, \dots, 0 \end{cases}$$

Asian Options (Laplace transform)

- For discretely monitored Asian option

$$\sum_{m=0}^{M} \tilde{S}_{t_m}^{(N)} = \sum_{m=0}^{M} g^{-1}(\tilde{X}_{t_m}^{(N)} + \rho f(v_{t_m}))$$
$$= \sum_{m=0}^{M} \zeta \circ \phi^{-1}(Y_{t_m}) = \sum_{m=0}^{M} h(Y_{t_m}) := \tilde{B}_M^{(N)},$$

- Let $v_d(M,k) = \mathbb{E}_{i,x_k}[(k - \tilde{B}_M^{(N)})^+]$, then

$$V_d(M,K) = \frac{e^{-rT}}{M+1} v_d(M,(M+1)K)$$

which is found upon inverting the Laplace transform:

$$\int_0^\infty e^{-\theta k} v_d(M,k)dk = \frac{\mathbf{e}_{i,x_k} \cdot (e^{-\theta \mathbf{D}} \mathbf{P}(\Delta))^M e^{-\theta \mathbf{D}} \cdot \mathbf{1}}{\theta^2}.$$

Occupation Time Derivatives

- Define continuous and discrete occupation time for barrier L:

$$\tau_T(L) := \int_0^T \mathbf{1}\{S_u \leq L\} du, \quad \tau_M(L) := \sum_{m=0}^{M} \mathbf{1}\{S_{t_m} \leq L\}$$

- Value of proportional payoff (with $\rho \geq 0$):

$$C_c(\kappa, T) = e^{-rT}\mathbb{E}[e^{-\rho\tau_T(L)}(S_T - e^{-\kappa})^+].$$

- Closed-form approximation of Laplace Transform:

$$\int_{-\infty}^{\infty} e^{-\phi\kappa} C_c(\kappa, T) d\kappa \approx \frac{e^{-rT}}{\phi(1+\phi)} \mathbf{e}_{i,x_k} \cdot e^{(\mathbf{G}-\tilde{\mathbf{D}})T} \tilde{V}_\phi(\mathbf{y}), \quad (4)$$

- (Note): $\tilde{V}_\phi(\cdot) := (\zeta \circ \phi^{-1}(\cdot))^{1+\phi}$, $\mathbf{y} := (1, 2, \ldots, N \cdot m_0)^T$, and $\tilde{\mathbf{D}} = (\tilde{d}_{nn})_{Nm_0 \times Nm_0}$ is a diagonal matrix $\tilde{d}_{nn} = \tilde{H}(n) = \rho \cdot \mathbb{1}\{\zeta \circ \phi^{-1}(n) \leq L\}$ for $n \in \mathcal{S}_Y$

- By the continuity of $g^{-1}(\cdot)$, it follows from Proposition 3 and the continuous mapping theorem that

$$\tilde{S}_T^{(N)} \Longrightarrow S_T, \quad \text{as} \quad m_0, N \to \infty,$$

- Again by the continuous mapping theorem we have $h(\tilde{S}_{t_m}^{(N)}) \Longrightarrow h(S_{t_m})$, for any continuous function h.
- Hence the Theorem 9 of Song et al. (2013)[10] implies that value of discretely monitored barrier/Bermudan/Asian options written on $h(\tilde{S}_{t_m}^{(N)})$ will converge to those written on $h(S_{t_m})$.

[10] Song, Q., Yin, G. and Zhang, Q., 2013. Weak convergence methods for approximation of the evaluation of path-dependent functionals. SIAM Journal on Control and Optimization, 51(5), pp.4189-4210.

SABR

- Recall the classical SABR model of Hagan et al (2002)[11]:

$$\begin{cases} dS_t = v_t S_t^{\beta} dW_t^{(1)}, \\ dv_t = \alpha v_t dW_t^{(2)}, \end{cases}$$

- Combines CEV local volatility S_t^{β} with GBM volatility process
- Auxiliary process:

$$\tilde{X}_t := g(\tilde{S}_t) - \rho f(v_{\alpha(t)}) = (\tilde{S}_t)^{1-\beta}/(1-\beta) - \rho v_{\alpha(t)}/\alpha,$$

- Dynamics

$$d\tilde{X}_t = \left(-\frac{\beta}{2(1-\beta)} \frac{v_{\alpha(t)}^2}{(\tilde{X}_t + \rho v_{\alpha(t)}/\alpha)}\right) dt + \sqrt{1-\rho^2} v_{\alpha(t)} dW_t^*$$

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[11] Hagan, P.S., Kumar, D., Lesniewski, A.S. and Woodward, D.E., 2002. Managing smile risk. The Best of Wilmott, 1, pp.249-296.

Extensions

- Given a local volatility component $\Gamma(S_t)$, we can "mix-and-match" our favorite variance processes to improve calibration in targeted markets
- E.g. variations of SABR include Heston-SABR, λ-SABR, etc., and many more variance dynamics can be applied
- Moreover, as long as the Lamperti transform $g(S_t)$ can be derived, monotonicity guarantees the existence of $g^{-1}(\cdot)$
- Hence, even if $g^{-1}(\cdot)$ is not available in closed form, we can easily invert numerically for the required values along our grid
- E.g. Hyp-Hyp model of Jackel and Kahl (2007)[12]

[12] Jackel, P. and Kahl, C., 2008. Hyp hyp hooray. Wilmott Magazine, 34, pp.70-81.

SABR Numerics (European): Case $\rho = 0$

		K						
	Δt	0.02	0.04	0.05	0.06	0.08	0.10	CPU(sec.)
Euler	1/400	0.0472	0.0429	0.0409	0.0389	0.0352	0.0318	49.4
	1/800	0.0463	0.0420	0.0399	0.0380	0.0343	0.0309	99.1
	1/1600	0.0453	0.0411	0.0391	0.0372	0.0336	0.0303	194.0
Low-bias	1/4	0.0461	0.0419	0.0399	0.0379	0.0343	0.0310	78.4
	1/8	0.0460	0.0418	0.0398	0.0378	0.0342	0.0308	175.8
Exact Sim	-	0.0457	0.0416	0.0396	0.0377	0.0341	0.0308	98.3
FDM	-	0.0456	0.0414	0.0394	0.0375	0.0339	0.0306	-
CTMC	-	0.0456	0.0414	0.0394	0.0375	0.0339	0.0306	1.8

Table: European call options. Parameters: $S_0 = 0.05, T = 1.0, \rho = 0, \beta = 0.30, \alpha = 0.60, v_0 = 0.40$. The Euler, Low-bias, Exact Sim, and FDM rows are reported in Song (2013): "*Essays on computational methods in financial engineering (Doctoral dissertation)*". Exact simulation results are obtained using $100,000$ sample paths.

SABR Numerics (European): Case $\rho \neq 0$

K	Exact simulation	StdErr	95CI	Expansion	CTMC
0.045	0.008789	1.30E-06	[0.008787 , 0.008792]	0.008788	0.008787
0.046	0.008204	1.34E-06	[0.008201 , 0.008206]	0.008202	0.008201
0.047	0.007644	1.37E-06	[0.007642 , 0.007647]	0.007643	0.007642
0.048	0.007111	1.39E-06	[0.007109 , 0.007114]	0.007110	0.007109
0.049	0.006605	1.41E-06	[0.006602 , 0.006608]	0.006603	0.006603
0.050	0.006124	1.43E-06	[0.006121 , 0.006127]	0.006123	0.006122
0.051	0.005669	1.44E-06	[0.005667 , 0.005672]	0.005668	0.005667
0.052	0.005240	1.45E-06	[0.005237 , 0.005243]	0.005238	0.005237
0.053	0.004835	1.45E-06	[0.004832 , 0.004838]	0.004833	0.004833
0.054	0.004454	1.44E-06	[0.004451 , 0.004457]	0.004453	0.004452
0.055	0.004096	1.44E-06	[0.004094 , 0.004099]	0.004095	0.004094

Table: European call option under SABR model. Parameters: $S_0 = 0.05, T = 1, \rho = -0.25, \beta = 0.55, \alpha = 0.03, v_0 = 0.08, r = 0$. The expansion column is obtained using expansion formulas in Hagan et al (2002) and Oblój (2008). Exact simulation results obtained using $10,240,000$ sample paths.

SABR Numerics (American and Barrier)

	K	1.00	1.02	1.04	1.06	1.08	1.10	1.12	1.14	1.16	1.18	1.20
American	LSM	0.0424	0.0494	0.0574	0.0659	0.0749	0.0852	0.0959	0.1074	0.1199	0.1326	0.1459
	CTMC	0.0424	0.0494	0.0572	0.0658	0.0751	0.0851	0.0959	0.1074	0.1196	0.1326	0.1461
Barrier	MC	0.0394	0.0463	0.0541	0.0624	0.0715	0.0814	0.0921	0.1034	0.1155	0.1283	0.1417
	CTMC	0.0396	0.0465	0.0542	0.0626	0.0717	0.0816	0.0923	0.1036	0.1157	0.1285	0.1419

Table: American and down-and-out barrier put options under the SABR model. Parameters:
$S_0 = 1.10, T = 1, \rho = -0.40, \beta = 0.70, \alpha = 0.08, v_0 = 0.20, r = 0.0$.
LSM is computed using 10^7 sample paths. Barrier option: $L = 0.6$ with $M = 250$ monitoring dates.

SABR Numerics (Asian)

M	K	0.90	0.95	1.00	1.05	1.10
250	MC	0.0059	0.0156	0.0345	0.0647	0.1048
250	CTMC	0.0058	0.0154	0.0344	0.0646	0.1047
50	MC	0.0058	0.0155	0.0344	0.0646	0.1048
50	CTMC	0.0057	0.0153	0.0342	0.0644	0.1046
12	MC	0.0056	0.0150	0.0338	0.0641	0.1045
12	CTMC	0.0055	0.0149	0.0337	0.0640	0.1044

Table: Discretely monitored arithmetic Asian put options under the SABR model. Parameters:
$S_0 = 1, r = 0, T = 1, \beta = 0.5, \alpha = 0.3, v_0 = 0.15, \rho = -0.5.$

Heston-SABR Numerics (Euro, American)

K/S_0	70%	80%	90%	100%	110%	120%	130%
MS-Eur Call	3.2066	2.2426	1.3108	0.5395	0.1276	0.0122	0.0003
CTMC-Eur Call	3.2069	2.2405	1.3101	0.5457	0.1281	0.0138	0.0007
LSM-Amer Put	0.0002	0.0042	0.0472	0.2802	0.9937	1.9930	2.9922
CTMC-Amer Put	0.0002	0.0044	0.0473	0.2813	0.9999	1.9999	2.9999

Table: Options under Heston-SABR . Parameters: $S_0 = 10$, $T = 1.0, \rho = -0.75, \alpha = 0.15, v_0 = 0.04, \theta = 0.035, \eta = 4, \beta = 0.70$, interest rate $r = 0.03$. LSM is computed using 10^7 sample paths.

Quadratic SLV Numerics (Euro, Barrier, American, Asian)

	K	8.0	9.0	9.5	10.0	10.5	11.0	12.0
European Call	MC	2.0003	1.0032	0.5319	0.1761	0.0251	0.0010	0.0000
	CTMC	2.0002	1.0035	0.5327	0.1759	0.0262	0.0017	0.0000
Down-Out-Call	MC	1.6281	0.8791	0.5046	0.1745	0.0251	0.0010	0.0000
	CTMC	1.6337	0.8813	0.5051	0.1742	0.0251	0.0011	0.0000
American Put	LSM	0.0000	0.0028	0.0315	0.1757	0.5250	1.0010	2.0000
	CTMC	0.0000	0.0033	0.0325	0.1757	0.5260	1.0015	1.9999
Asian Put	MC	0.0000	0.0000	0.0032	0.1014	0.5018	1.0002	2.0002
	CTMC	0.0000	0.0000	0.0039	0.1032	0.5017	0.9989	1.9991

Table: Quadratic SLV model: $r = 0, S_0 = 10$, $T = 0.5, v_0 = 0.035, \eta = 4, \theta = 0.03, \rho = -0.7, a = 0.02, b = 0.05, c = 1$. Down-and-out barrier $B = 9.5$, and $M = 500$. Asian put with $M = 12$.

Thank You

Q & A