Developments in ODE, SDE, PDE, PIDE and Other Analytical Approaches and their Applications to Modern Physics and Quantitative Finance

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ABSTRACT

We discuss certain latest developments in methodology and approaches to solve ordinary differential equations (ODE), stochastic differential equations (SDE), partial differential equations (PDE), partial integro-differential equations (PIDE) and related objects analytically.

These approaches are used in both Modern Physics and Quantitative Finance both theoretically and in practical applications. An additional advantage is that the approaches developed in Physics could be often applied in Quantitative Finance & vice versa.

In our presentation, we will show that these analytical methodologies are making both research and its implementation in both Physics and Quantitative Finance much more efficient.

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Advantages of Analytical Approaches to Solving ODE, SDE, PDE and PIDE

- Though ODE, SDE, PDE and PIDE are primarily solved using numerical methods in both academia and industry today, analytical approaches have their profound advantages
- The following are these advantages:
  - Analytical approaches are precise while numerical methods are approximate
  - Analytical approaches are much faster than numerical methods
  - Analytical approaches are producing the same results by definition while numerical methods might produce different results when run twice (e.g., Monte Carlo); this consistency is a very important advantage
  - Analytical approaches produce much better and much faster sensitivities than numerical methods (e.g., cross-Gamma effect proves the point)
  - The results of analytical approaches are formulas that, with some assumptions, represent laws of nature, psychology or economy (e.g., Einstein's $E = mc^2$ or a Black-Scholes formula) by giving an explicit dependency of the results on its underlying parameters while numerical methods operate like black boxes where one should put the inputs to get the outputs out of them
- The disadvantage of the analytical approaches compared to numerical methods is that they are often not trivial to obtain, so they are currently not known for many ODE, SDE, PDE and PIDE, and numerical methods are still extensively being used in these cases
- Our research is geared towards development of the non-trivial analytical approaches
- The ultimate goal would be to develop a general theory of analytical solution for them, but let's start with the general approach for Physics and Quantitative Finance

Application of these Analytical Methodologies to Modern Physics and Quantitative Finance – General Approach

- For Physics, the analytical methodologies below are being applied to:
  - The Heat Equation – its1-dimensional formulation is

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, k > 0$$

  - The Wave Equation – its 1-dimensional formulation is

$$\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2}$$

  - Schodinger and other PDEs and PIDEs of Particle Physics and Quantum Mechanics including Generalized Gross-Pitaevskii Equations (GGPEs) below
  - We are applying the analytical methodologies below to solve homogeneous and non-homogeneous Schrodinger equations with various boundary conditions
  - One of the perspective directions in Modern Physics lately is to study interaction of various particles in 2-dimensional and 3-dimensional media; GGPEs in 2-dimensional media for 2 wave functions could be written in the form:

$$u = \frac{1}{k_4}\frac{\partial v}{\partial t} + k_5\frac{\partial^2 v}{\partial {x_1}^2} + k_6\frac{\partial^2 v}{\partial {x_2}^2} - k_7(x_1, x_2)u - k_8 \,|\, v \,|^2\, v$$

$$\frac{\partial u}{\partial t} + k_1 \frac{\partial^2 u}{\partial {x_1}^2} + k_2 \frac{\partial^2 u}{\partial {x_2}^2} + k_3 u = k_4 v,$$

- These non-homogeneous PDEs are derived by varying the energy functional with respect to the wave functions *u* and *v*. If we substitute *u* from the second equation to the first equation (or, visa versa, *v* from the first equation to the second equation), we will get a complex 4-dimensional nonlinear PDE, so we solved the first non-homogeneous linear PDE closed-form, expressed this way *u* in terms of *v*, substituted *u* in the second PDE to obtain the following PIDE:

$$\frac{k_4}{2k_1k_2}\int_0^t\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} v(x,y,\tau)e^{-k_3(t-\tau)-\frac{(x_1-x)^2}{4k_1^2(t-\tau)}-\frac{(x_2-y)^2}{4k_2^2(t-\tau)}}dxdy\frac{d\tau}{t-\tau} = \frac{1}{k_4}\frac{\partial v}{\partial t} + k_5\frac{\partial^2 v}{\partial {x_1}^2} + k_6\frac{\partial^2 v}{\partial {x_2}^2} - k_7(x_1,x_2)u - k_8\,|v|^2\,v$$

We have solved GGPEs via a decomposition of both wave functions into the orthonormal bases after going to polar coordinates numerically (though we computed 6 integrals needed closed-form) and now are finalizing working on the analytical approximation solution of the PIDE above using the analytical methodologies discussed below; the nearest goal is to find an analytical approximation (numerico-analytical) solution for this PIDE and then go to the analytical solution with further theoretical developments, per below

- It could be proved, similarly to the above, that solving a PIDE is equivalent to solving a system of non-homogeneous PDEs (one could also trivially decompose a PIDE into a PDE and an integral equation or a convolution); for numerical solution purposes, it is probably better to present a PIDE in terms of a system of non-homogeneous PDEs and then to solve them numerically; for analytical approaches, it is probably better to deduce a PIDE from a system of non-homogeneous PDEs, like we did above, and then solve the PIDE analytically using the analytical approaches, per below

- Another approach in Physic to the solution of such PDEs and PIDEs is to use a parameter-driven solution and then find an energy via optimization closed-form
- And the third approach in Physics is to deduce PIDEs to PDEs and then PDEs to ODEs via explicit harmonics as functions of time (say, $u(x,y,t) = v(x,y)e^{i\omega t}$) and then solve the ODE either trivially (if the ODE is with constant coefficients) or using the methodologies below (if the ODE has variable coefficients; e.g., integral transforms)
- For Quantitative Finance, let's start with a famous Black-Scholes PDE (we are putting it in its generalized form; from now on, we are treating PDEs as partial cases of PIDEs where λ = 0, per below)

$$\frac{\partial V}{\partial t} + \mu_t S \frac{\partial V}{\partial S} + \frac{\sigma_t^2 S^2}{2} \frac{\partial^2 V}{\partial S^2} = r_t V$$

(here *S* is an Underlying Asset Value, *t* is Valuation Time, *V(S, t)* is a Derivative's Price, *μ* is a term-structured drift, *σ* is a term-structured Implied Volatility and *r* is a term-structured risk-free interest rate), PDEs and PIDEs are widely applied to pricing and risking derivative securities and their portfolios (for the multi-dimensional derivative security or a portfolio of derivatives, the following PDE could be used:

$$\frac{\partial V}{\partial t} + \sum_{l=1}^{m} \mu_t^{(l)} S_l \frac{\partial V}{\partial S_l} + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m} \sigma_t^{(i)} \sigma_t^{(j)} \rho_t^{(ij)} S_i S_j \frac{\partial^2 V}{\partial S_i \partial S_j} = r_t V$$

where *S1,…, Sm* are Underlying Values for *m* Assets (possibly, from different Asset Classes) or Factors, *t* is Valuation Time, *V(S1,…, Sm, t)* is a Derivative's (Portfolio) Price, *μ* are term-structured drifts, *σ* are term-structured Implied Volatilities, *ρ* are term-structured Correlation coefficients, *r* is a term-structured risk-free interest rate)

- For both 1-dimensional derivatives, multi-dimensional derivatives and their portfolios, different and more complex PDEs and PIDEs could be used, as we shall see below
- We will show the examples of PDEs and PIDEs for structured and non-structured derivative products below, but the general approach is to solve them analytically by using one of the analytical approaches below, combination of them, by developing a novel approach, a new theory or a general theory (that is the ultimate goal)
- These analytical approaches could be also applied to the ODEs, PDEs and PIDEs that are used in high frequency trading, processing, modeling and data including parameter calibration, for example, to solve a Hamilton-Jacobi-Bellman (HJB) PDE
- These nonlinear and linear ODEs, PDEs and PIDEs, including HJB PDE, should be tackled similarly to the differential equations above and below, as well as using the most recent analytical approaches developed by PDE researchers including us
- So, let's give several examples of these analytical approaches that could also be applied to pricing products with recent developments

Developments in these Analytical Models and their Applications with Examples

- Let's give the examples, besides 2 PDEs above, with the approaches suggested
- To value spread options (or any options where the underlying could be positive or negative), the following Hull-White (normal with mean reversion) PDE is used:

$$\frac{\partial V}{\partial t} + \nu_t(\mu_t - S)\frac{\partial V}{\partial S} + \frac{\sigma_t^2}{2}\frac{\partial^2 V}{\partial S^2} = r_t V$$

(here *S* is an Underlying Asset Value, *t* is Valuation Time, *V(S, t)* is a Derivative's Price, *μ* is a term-structured Long-Term Mean, *v* is a term-structured Mean Reversion Speed, *σ* is a term-structured Implied Volatility, *r* is a term-structured risk-free interest rate); it is solved by using a non-integral transform that gets rid of mean reversion above and then using an integral Fourier transform and an inverse Fourier transform, per below

- If we want to value options on a lognormal asset with mean reversion (say, CDS spread without a jump, yield, realized variance, etc.), we would use a Black-Karasinski PDE

$$\frac{\partial V}{\partial t} + \nu_t(\mu_t - \ln S)S\frac{\partial V}{\partial S} + \frac{\sigma_t^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} = r_t V$$

(here *S* is an Underlying Asset Value, *t* is Valuation Time, *V(S, t)* is a Derivative's Price, *μ* is a term-structured Long-Term Mean, *v* is a term-structured Mean Reversion Speed, *σ* is a term-structured

Implied Volatility, *r* is a term-structured risk-free interest rate); it is solved by using u = ln S and a non-integral transform that gets rid of mean reversion above and then using a Fourier transform and an inverse Fourier transform, per below

- It's important to note that in Black-Karasinski PDE and SDE, a *natural logarithm* of the lognormal asset reverts to the mean rather than the asset *itself*; if the lognormal asset *itself* reverts to the mean in the SDE, a corresponding PDE could be reduced to

$$\frac{\partial V}{\partial t} + \frac{\sigma_t^2 (S + m(t))^2}{2} \frac{\partial^2 V}{\partial S^2} = r_t V$$

Here *m'(t)* is a drift in the corresponding SDE; with a further functional change, we are obtaining an Andreasen-Vecer PDE for Asian Options on discrete average as

$$\frac{\partial g}{\partial t} + \frac{\sigma_t^2 (S + m(t))^2}{2} \frac{\partial^2 g}{\partial S^2} = 0$$

There is no closed-form solution for this PDE; we established an equivalence of this PDE for Asian Options on discrete average and a lognormal mean-reverting *itself* asset SDE and now working on the analytical solution of this PDE.

- To value spread options (or any options where the underlying could be positive or negative), the following Cox-Ingersoll-Ross (CIR) (normal with mean reversion and square root of the underlying-dependent volatility) PDE could be used as well:

$$\frac{\partial V}{\partial t} + \nu_t (\mu_t - S) \frac{\partial V}{\partial S} + \frac{\sigma_t^2 S}{2} \frac{\partial^2 V}{\partial S^2} = r_t V$$

(here *S* is an Underlying Asset Value, *t* is Valuation Time, *V(S, t)* is a Derivative's Price, *μ* is a term-structured Long-Term Mean, *v* is a term-structured Mean Reversion Speed, *σ* is a term-structured Implied Volatility, *r* is a term-structured risk-free interest rate); it is solved by using a non-integral transform that gets rid of mean reversion above and then using an integral Fourier transform (though it is harder than for the Black-Scholes PDE, Hull-White PDE or Black-Karasinski PDE) and an inverse Fourier transform, per below

- We could expand CIR to a lognormal case, with the same notations and the same analytical solution approach as for CIR + u = ln S, to arrive at the following PDE:

$$\frac{\partial V}{\partial t} + \nu_t(\mu_t - \ln S)S\frac{\partial V}{\partial S} + \frac{\sigma_t^2 S^2 \ln S}{2}\frac{\partial^2 V}{\partial S^2} = r_t V$$

- We are accounting for Liquidity in such PDEs generalizing it to the following PDE:

$$\frac{\partial V}{\partial t} + A(S,t)\frac{\partial V}{\partial S} + \frac{\sigma^2(S,t)}{2\left(1 - \frac{\frac{\partial^2 P}{\partial S^2}}{L(S,t)}\right)^2}\frac{\partial^2 V}{\partial S^2} = r_t V$$

- If Liquidity is perfect (L is ∞ in the equation above), this PDE becomes the prior PDEs
- We could also account for Liquidity in a Black-Karasinski PDE (a good idea from the financial crisis, as all the PDEs above assume a perfect liquidity that is definitely not the case during the crisis) using the following nonlinear PDE that is generalizing the above:

$$\frac{\partial V}{\partial t} + \nu_t(\mu_t - \ln S)S\frac{\partial V}{\partial S} + \frac{\sigma_t^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} + \sum_{k=1}^{n} c_t^{(k)} S^k f_k\left(\frac{\partial^2 V}{\partial S^2}\right) = r_t V$$

(here *S* is an Underlying Asset Value, *t* is Valuation Time, *V(S, t)* is a Derivative's Price, *μ* is a term-structured Long-Term Mean, *v* is a term-structured Mean Reversion Speed, *σ* is a term-structured Implied Volatility, *c(k)* are term-structured weights that depend on liquidity (stochastic in future models), *fk()* are, generally speaking, nonlinear (and not necessarily differentiable) functions of Gamma, *r* is a term-structured risk-free interest rate); for some partial cases of *fk()*, using the ideas above, we obtained closed-form solutions for European options, though corresponding PDEs, similarly to MBS PDE that will be discussed below, proved to be reduced to non-homogeneous ones, along with analytical approximations for non-European options; for other cases of *fk()*, we obtained analytical approximations as well; other authors suggested their own analytical approaches to solve such PDEs

- For multi-dimensional (multi-asset and hybrid) derivatives and portfolios of one-dimensional (one-factor) and multi-dimensional (multi-factor) derivatives, the nonlinear PDE above will be generalized to the following multi-dimensional nonlinear PDE:

$$\frac{\partial V}{\partial t} + \sum_{l=1}^{m} \nu_t^{(l)} (\mu_t^{(l)} - \ln S_l) S_l \frac{\partial V}{\partial S_l} + \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} \sigma_t^{(i)} \sigma_t^{(j)} \rho_t^{(ij)} S_i S_j \frac{\partial^2 V}{\partial S_i \partial S_j} + \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} d_t^{(i)} d_t^{(j)} \sigma_t^{(i)} \sigma_t^{(j)} \rho_t^{(ij)} (S_i S_j)^2 \frac{\partial^2 V}{\partial S_i^{\,2}} \frac{\partial^2 V}{\partial S_j^{\,2}} \frac{\partial^2 V}{\partial S_i \partial S_j} + \sum_{l=1}^{m} \sum_{k=1}^{n} c_t^{(l)(k)} S_l^{\,k} f_k\left(\frac{\partial^2 V}{\partial S_l^{\,2}}\right) = r_t V$$

(here *S1,…, Sm* are Underlying Values for *m* Assets (possibly, from different Asset Classes) or Factors, *t* is Valuation Time, *V(S1,…, Sm, t)* is a Derivative's (Portfolio) Price, *μ* are term-structured Long-Term Means, *v* are term-structured Mean Reversion Speeds, *σ* are term-structured Implied Volatilities, *ρ* are term-structured Correlation coefficients, *d(i)* and *c(l)(k)* are term-structured weights that depend on liquidity (stochastic in future models), *fk()* are, generally speaking, nonlinear (and not necessarily differentiable) functions of Gamma, *r* is a term-structured risk-free interest rate; for some partial cases of *fk()*, using the ideas above and generalizing a one-dimensional case, we obtained closed-form solutions for European options, though corresponding PDEs, similarly to MBS PDE that will be discussed below, proved to be reduced to non-homogeneous ones, along with analytical approximations for non-European optionality and other cases of *fk()*

- Such a generalization (with possible switch for some assets from the lognormal mean-reverted dynamics above to a normal mean-reverted dynamics) could be done for all the multi-asset and hybrid products described below, as well as exotic one-dimensional ones, so let's briefly look at them
- The following is a 2-dimensional Paul Wilmott PDE for a Convertible Bond:

$$\frac{\partial V}{\partial t} + \frac{\sigma_t^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} + \rho_t \sigma_t w_t S\frac{\partial^2 V}{\partial S\,\partial r} + \frac{w_t^{\,2}}{2}\frac{\partial^2 V}{\partial r^2} + (r - d_t)S\frac{\partial V}{\partial S} + (a_t - b_t r)\frac{\partial V}{\partial r} = rV$$

(here V is a price of a Convertible Bond, *t* is Valuation Time, S is a Stock Price Per Share, *r* is a Credit-Risky Interest Rate, *d* is a Dividend Yield, *a* and *b* are mean reversion parameters, σ is a Stock Implied Volatility, *w* is an Interest Rate Implied Volatility and ρ is a correlation coefficient between the Stock and the Interest Rate); this PDE and its partial cases should be valued using Fourier and other integral and non-integral transforms to get an analytical solution for European Options

- The following is a multi-dimensional non-homogeneous PDE that is valuing a key element of a Mortgage-Backed Security (MBS), per below:

$$\frac{\partial U_t}{\partial t} + \frac{\sigma_S^2 S^2}{2}\frac{\partial^2 U_t}{\partial S^2} + \frac{\sigma_O^2}{2}\frac{\partial^2 U_t}{\partial O^2} + \rho\sigma_S\sigma_O S\ \frac{\partial^2 U_t}{\partial S \partial O} + \mu_S S\frac{\partial U_t}{\partial S} + (a_O - b_O O)\frac{\partial U_t}{\partial O} =$$

$$rU_t - (PTR - r)a_t$$

where $U(S, O, T) = \frac{P}{12} max\left(O - \left[PTR - r\right], 0\right)$

- Here U is a price of an MBS option portion, *t* is Valuation Time, S is a Prepayment Speed (PSA), *r* is a risk-free interest rate, *O* is an Option-Adjusted Spread (OAS), σ_S is a PSA Implied Volatility, σ_O is an OAS Implied Volatility, a_t is an Outstanding Mortgage Balance, μ_S is a PSA drift, ρ is a correlation coefficient between the PSA and the OAS, a_O and b_O are OAS mean reversion parameters (a_O/b_O is a Long-Term Mean and b_O is a Mean Reversion Speed), PTR is a Pass-Through (Mortgage) Rate, P is a Par Amount and T is maturity; this PDE is solved closed-form via our analytical approaches
- We are solving the PDE using integral and non-integral transforms; we obtained an analytical approximation solution and just arrived at a closed-form solution, similarly to solving the first GGPE in Particle Physics applications above, for this PDE and its generalizations using approaches below & approaches to PDE solution in Physics above
- Many more examples could be given for PDEs (Asian Integral Average Options PDE, a popular Heston 2-dimensional PDE), but let's give an example of a (Merton-76) PIDE:

$$\frac{\partial V}{\partial t} + (r_t - d_t)S\frac{\partial V}{\partial S} + \frac{\sigma_t^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} + \lambda\int_{-\infty}^{+\infty}[\mathrm{V}(\mathrm{Se}^{\mathrm{x}}, t) - V - S(\mathrm{e}^{\mathrm{x}} - 1)\frac{\partial V}{\partial S}]dN([x - \ln(1 + j) + \gamma^2/2]/\gamma) = r_t V$$

(here V is a price of an Option, *t* is Valuation Time, S is a Stock Price Per Share, *r* is a Risk-Free term-structured Interest Rate, *d* is a term-structured Dividend Yield, σ is a Stock Term-Structured Implied Volatility, λ is a jump intensity, *j* is a relative jump size and γ is a Volatility of a jump size); this PIDE should be valued using Fourier and other integral and non-integral transforms, as well as Operator Theory, per below, to get an analytical solution for European Options; here, we are starting by using the property of a convolution that the Fourier-image of a convolution is a product of Fourier-images of both functions to the convolution in the PIDE above, simultaneously with applying this Fourier transform to the PDE part of the PIDE (after the change of variable u = ln S); this PIDE is simpler than a PIDE above that we derived from GGPEs, as

- It is linear while our PIDE derived from GGPEs is nonlinear
- The derivative pricing function in the PIDE above is real-valued while our wave functions in Physics are complex-valued
- There is no integration over time while there is one in Physics (it will be integration over time if we generalize our MBS PDE to PIDE by assuming jump-diffusion for OAS, PSA and possibly interest rates)
- There are more dimensions in the Physics PIDE (though the above Quantitative Finance PIDE could be generalized to more than one spatial dimension as well)

- The most recent developments in these analytical approaches that should result in our forthcoming publications this year include:
    - An article in Modern Physics that includes, among other analytical results, a generalization of an infinite series Cauchy product theorem to the multi-dimensional and infinite dimensional (operator) cases
    - A paper that should include Model Risk Analytics methodologies that are using ODEs, PDEs, PIDEs and operator theory, among other analytical and statistical approaches
    - Possible publication in the area of non-homogeneous PDEs that will use an analytical approach to a solution of a system of such PDEs to price Structured Products and Exotic Derivatives
    - A possible paper that will focus on application of the above analytical approaches to Risk Analytics of major financial risk types – Market, Credit, Liquidity, Operational and Model Risks
    - A probable paper in the application of these analytical approaches in Modern Advanced Physics
- So, let’s discuss the analytical approaches to solving PDEs and PIDE in the framework of the general approach and all the analytical approaches discussed above (used and developed in both Modern Physics and Quantitative Finance) and with the purpose of further development of new approaches and potentially a new theory and a general theory of these approaches

Overview of Analytical Approaches to Solving ODEs, PDEs and PIDEs

So, the following is an overview of these PDE approaches:

- Integral Transforms:
  - Fourier transform – is used to solve ODEs, PDEs and PIDEs in Physics and Finance by its application to, say, a normally distributed variable, calculating a Fourier-image using the initial or terminal condition and then applying an inverse Fourier transform to come up with the solution (is a challenge in CIR and even a bigger challenge in Heston (called "Heston trap"); for PIDEs, in addition to the above, a Fourier transform is applied to a convolution to get a product of Fourier images (as mentioned above), though the inverse transform step is a challenge and requires additional work (operators, etc.)
  - Laplace transform – is used to solve ODEs, PDEs and PIDEs in Physics and Finance by its application to a positive variable (usually, time); the biggest challenge, though, is to get a solution after the inverse Laplace transform calculation, as the integral has complex boundaries and is not straightforward to assess
  - Fourier-Wiener-Feinman transform – very useful in both Physics and Finance, as it has a clear inverse from the same family, a clear product from the same family and other properties; it could be used

to solve ODEs, PDEs and PIDEs not only with constant and term-structured coefficients, but also with linear and some non-linear variable coefficients, so we are continuing our research for it

    - <u>Other integral transforms</u> – there are other integral transforms used mostly in Physics, but we think that the perspective direction is to gear the research to creating (and further improving existing) integral transforms that are capable to solve linear ODEs, PDEs & PIDEs with variable coefficients and nonlinear PDEs and PIDEs

- <u>Non-Integral Transforms</u> (d'Alembert and beyond) – French philosopher, Encyclopedist and mathematician Jean-Baptiste le Rond d'Alembert applied the first non-integral transform to solve a one-dimensional wave equation in the 18th century; this approach is being extended to many dimensions, other equations in Finance and Physics and to a nonlinear paradigm; the approach is quite powerful together with the above, especially to one-dimensional and multi-dimensional wave equations and their generalizations
- Operator Theory:
    - <u>Integral Operators</u> – they are used in the form of integral transforms, per above, & several other forms
    - <u>Differential Operators</u> – a very powerful and perspective tool for the solution of PDEs and PIDEs in Finance and Physics; they could be used in many forms and via many approaches by themselves, as well as participating in more general approaches and solutions

    - General philosophy – operator theory is a very powerful tool to be used to solve linear and nonlinear ODEs, PDEs and PIDEs, but the general philosophy is to use it in conjunction with the approaches above and/or the part of a new approach or a general theory
  - Connection between Integral / Non-Integral Transforms and Operator Theory / Application of Operators as an approach to solve PDEs / PIDEs with finite boundary conditions in Physics and Math Finance – our recent research shows that Integral / Non-Integral Transforms and their inverses could be expressed via operators above and vice versa; this allows to more effectively solve PDEs and PIDEs with finite boundary conditions in both Physics (when heat or field characteristics are studied in a finite 1-dimensional or multi-dimensional volume) and Mathematical Finance (for certain exotic options, structured trades and their portfolios)
  - Other Functional Analysis methodologies (e.g., Integral Equations) – other Functional Analysis methodologies, including Integral Equations, functionals, norms and scalar products applications, etc. Again, these methodologies should be applied to solving linear and nonlinear ODEs, PDEs and PIDEs in conjunction with the approaches above and/or the part of a new approach or a general theory
  - Other analytical approaches are being developed by us and other researchers and should be developed further. One of the main goals of this research is to find effective analytical solutions for linear and nonlinear ODEs, PDEs and PIDEs and develop their theory

Conclusion and Outline of Further Research in this Subject

- We presented an overview of analytical approaches to the solution of ODEs, PDEs and PIDEs in light of their application to Physics and pricing and risking derivative securities and their portfolios and gave the examples of structured and non-structured products
- These analytical approaches are especially important in Risk Management in Finance (with a great potential on the Desks and in Model Validation) and in Physics, especially in rapidly developing areas of Particle Physics, Solid State Physics and Spectroscopy, as well as Material Science
- Further research in the area of analytical approaches to solving linear and nonlinear ODEs, PDEs and PIDEs should be focused on:
  - Further development in the area of Integral and Non-Integral Transforms
  - Further development in the area of Operator Theory
  - Further developing Functional Analysis methodologies
  - Further development of other Analytical approaches to solving ODEs, PDEs and PIDEs, as well as the new theories that will enable us to analytically solve <u>certain classes</u> of linear and nonlinear ODEs, PDEs and PIDEs and then to build a general theory of analytical solution of all the PDEs and PIDEs; this should also include exploring a connection between differential forms and linear and non-linear dynamic systems and use analytical approaches for solutions for these dynamic systems to optimize usage of differential forms and tensors in modern field theory in Modern Physics and Modern Quantitative Finance

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