A stochastic model for order book dynamics

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(with Rama Cont and Rishi Talreja)

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Motivation

Two major types of market mechanisms:

1. Quote driven (e.g. NYSE specialist)
2. Order driven (e.g. Arca, Instinet, Tokyo stock exchange)
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- Two approaches to order driven markets:
  1. Markets are made up of many rational individuals acting in their own best interest
     (e.g. Parlour (1998), Foucault et al. (2005), Rosu (2008))
  2. The collective behavior of these rational people can be reproduced by modeling non strategic traders
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- These models seem challenging to estimate
Model objectives

- To predict short term price behavior
  1. Given the current order book
  2. Given statistics on the order flow
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- We should focus on computing conditional probabilities
- Our model should be easy to estimate
The order book

- What is an order book?
- Modeling the order book
Outline

1 The order book
   • What is an order book?
   • Modeling the order book

2 Estimation
   • The rates of market orders, limit orders and cancellations
   • Comparing data to simulation
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   - Probability of executing a limit order before the price moves
   - Probability of making the spread
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4. Conclusion
A market order

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A limit order
A limit order
A cancellation

## Order Book

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Continuous-time Markov chain $X_t \equiv (X^1_t, \ldots, X^n_t)$, where $|X^i_t|$ is the number of limit orders in the book at price $i$. 
Notation

- Continuous-time Markov chain \( X_t \equiv (X_t^1, \ldots, X_t^n) \), where \( |X_t^i| \) is the number of limit orders in the book at price \( i \).
- If \( X_t^i < 0 \) then there are \(-X_t^i\) bid orders at price \( i\); if \( X_t^i > 0 \) then there are \( X_t^i \) ask orders at price \( i \).
Notation

- Continuous-time Markov chain $X_t \equiv (X_{t1}, \ldots, X_{tn})$, where $|X_{ti}|$ is the number of limit orders in the book at price $i$
- If $X_{ti} < 0$ then there are $-X_{ti}$ bid orders at price $i$; if $X_{ti} > 0$ then there are $X_{ti}$ ask orders at price $i$.
- The best ask process
  \[ p_A(t) = \inf \{i, \ X_{ti} > 0\}, \quad t \geq 0. \]
  The best bid process
  \[ p_B(t) = \sup \{i, \ X_{ti} < 0\}, \quad t \geq 0. \]
Assumptions

- Market buy (resp. sell) orders arrive at independent, exponential times with rate $\mu$.
- Limit buy (resp. sell) orders arrive at a distance of $i$ ticks from the opposite best quote at independent, exponential times with rate $\lambda(i)$.
- Cancellations of limit orders at a distance of $i$ ticks from the opposite best quote occur at a rate proportional to the number of outstanding orders: if the number of outstanding orders at that level is $x$ then the cancellation rate is $\theta(i)x$.
- The above events are mutually independent.
The transition rates

Let

\[ x^{i\pm 1} \equiv x \pm (0, \ldots, 1, \ldots, 0), \]

where the 1 in the vector on the right-hand side is in the \( i \)th component.

\[
\begin{align*}
  x \rightarrow x^{i-1} & \quad \text{with rate } \lambda(p_A(t) - i) \quad \text{for } i < p_A(t), \\
  x \rightarrow x^{i+1} & \quad \text{with rate } \lambda(i - p_B(t)) \quad \text{for } i > p_B(t), \\
  x \rightarrow x^{p_B(t)+1} & \quad \text{with rate } \mu \\
  x \rightarrow x^{p_A(t)-1} & \quad \text{with rate } \mu \\
  x \rightarrow x^{i+1} & \quad \text{with rate } \theta(p_A(t) - i)|x^i| \quad \text{for } i < p_A(t), \\
  x \rightarrow x^{i-1} & \quad \text{with rate } \theta(i - p_B(t))|x^i| \quad \text{for } i > p_A(t),
\end{align*}
\]
Ergodic property

Proposition

$X$ is an ergodic Markov process. In particular, $X$ has a proper stationary distribution.

We may compute time averages of various quantities in a simulation

- average shape of the order book
- volatility

and interpret them as expectations in the model.
The market order rate

- Number of seconds in 125 days of data
  \[ T = 125 \times 4.5 \times 60 \times 60 = 2,025,000 \text{s} \]
- Number of market orders in our data \( N_{\mu} = 64,777 \)
- Average size of market orders \( S_{\mu} = 8.84 \)
- Average size of limit orders \( S_{\lambda} = 14.27 \)
The market order rate

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- The market order rate
  \[
  \hat{\mu} = \frac{N_\mu S_\mu}{T S_\lambda} = 0.0198
  \]
  orders per second
The limit order rate function

- Number of limit orders at distances 1-5 from the best opposite quote in our data
  \[ N_\lambda = [71,884 \quad 50,851 \quad 36,825 \quad 29,567 \quad 25,831] \]
The limit order rate function

- Number of limit orders at distances 1-5 from the best opposite quote in our data
  \[ N_{\lambda} = [71, 884 \quad 50, 851 \quad 36, 825 \quad 29, 567 \quad 25, 831] \]
- Limit order rate function for \( 1 \leq d \leq 5 \):
  \[ \hat{\lambda}(d) = \frac{N_{\lambda}(d)}{T} \]

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  \]
  orders per second.
- For \( d > 5 \) extrapolate with a power law function of the form
  \[
  \hat{\lambda}(d) = \frac{k}{d^\alpha}
  \]
  \( k \) and \( \alpha \) are obtained by minimizing the least square distance
  \[
  \min_{k,\alpha} \sum_{d=1}^{5} \left( \hat{\lambda}(d) - \frac{k}{d^\alpha} \right)^2.
  \]
The limit order rate function

![Graph showing the limit order rate function. The x-axis represents the distance from the opposite best quote in ticks, ranging from 1 to 10. The y-axis represents the limit order arrival rate, ranging from 0.005 to 0.04. The graph compares data and simulation results.](image-url)
The cancel rate function

- Number of cancellations in our data (distances 1-5 from the best opposite quote)

\[ N_\theta = [23, 829 \quad 37, 698 \quad 32, 370 \quad 27, 264 \quad 23, 218] \]
The cancel rate function

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  \[ N_\theta = [23,829 \quad 37,698 \quad 32,370 \quad 27,264 \quad 23,218] \]
- Average cancellation size \( S_\theta = 18.61 \)
- Average number of orders at a distance of \( d \) from best quote \( Q(d) \).
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- Average number of orders at a distance of \( d \) from best quote \( Q(d) \).
- Cancel rate function for \( 1 \leq d \leq 5 \):
  \[ \hat{\theta}(d) = \frac{N_\theta(d)}{TQ(d)} \frac{S_\theta}{S_\lambda} \]
- For \( d > 5 \), we let \( \hat{\theta}(d) = \hat{\theta}(5) \).
The cancel rate function

![Graph showing the cancel rate function with data and simulation lines. The x-axis represents the distance from the opposite best quote in ticks, ranging from 1 to 10, and the y-axis represents the cancelation rate per order, ranging from 0 to 0.014.]
Comparing movies

Comparing Sample paths

![Graph showing comparison between empirical and simulation paths of traded prices over a number of trades.](image)
Comparing average order book shape
Comparing one step ahead predictions

- In our model, we may compute the probability of a queue going up, when there are \( m \) orders in the queue, for \( 1 \leq d \leq 5 \), conditional on the best quotes not changing.

\[
p_{up}^d(m) = \frac{\hat{\lambda}(d)}{\hat{\theta}(d)m + \hat{\lambda}(d) + \hat{\mu}}
\]

for \( d = 1 \)

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• We can compare these probabilities to the empirical probabilities of these queues going up
Comparing one step ahead predictions

- 1 tick from opposite quote
- 2 ticks from opposite quote
- 3 ticks from opposite quote
- 4 ticks from opposite quote
- 5 ticks from opposite quote
Conditional probabilities of interest

- The probability that the midprice goes up before it goes down (spread=1)
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- The probability that the midprice goes up before it goes down (spread>1)
- The probability that an order at the bid executes before the ask queue disappears (spread=1)
- The probability that both a buy and a sell limit order execute before the best quotes move (spread=1)
Laplace transforms

- The two-sided Laplace transform

\[ \hat{f}(s) = E[e^{-sT}] = \int_{-\infty}^{\infty} e^{-st} f(t) \, dt \]

where \( f(t) \) is the pdf of a random variable \( T \)
Laplace transforms

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- The inverse

\[ f(t) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} e^{ts} \hat{f}(s) ds \]
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- We may compute the Laplace transform of many random variables of interest in our model

- Numerically computing the inverse is fast!
A birth death process

- For each price level, the number of orders is a birth death process with birth rate $\lambda$ and death rate $\mu_k = \mu + k\theta$. 
A birth death process

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- \( T^{i,i-1} \) - first time that the BD process goes from \( i \) to \( i - 1 \)
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- The Laplace transform of the first passage time

$$\hat{f}_{i,i-1}(s) = E[e^{-sT^{i,i-1}}]$$
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- A recurrence relation for $\hat{f}$

$$\hat{f}_{i,i-1}(s) = \frac{\mu_i}{\mu_i + \lambda + s} + \frac{\lambda}{\mu_i + \lambda + s} \hat{f}_{i+1,i}(s)\hat{f}_{i,i-1}(s)$$
First passage time of a birth death process

- The recurrence relation allows us to express the Laplace transform of the first passage time as a continued fraction

\[
\hat{f}_{i,i-1}(s) = -\frac{1}{\lambda} \Phi_k^{\infty} i \frac{-\lambda \mu_k}{\lambda + \mu_k + s}
\]
First passage time of a birth death process

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\[
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\]

- Continued fractions:

\[
\Phi_{k=1}^{\infty} \frac{a_k}{b_k} \equiv \lim_{n \to \infty} w_n
\]

where

\[
w_n = t_1 \circ t_2 \circ \cdots \circ t_n(0), \quad n \geq 1,
\]

and

\[
t_k(u) = \frac{a_k}{b_k + u}, \quad k \geq 1.
\]
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- Abate and Whitt (1999)
First passage time of a birth death process

- Let $\sigma_b$ denote the first-passage time to 0 of a BD process starting at $b$.
First passage time of a birth death process

- Let $\sigma_b$ denote the first-passage time to 0 of a BD process starting at $b$

\[ \sigma_b = \sigma_{b,b-1} + \sigma_{b-1,b-2} + \cdots + \sigma_{1,0} \]

where $\sigma_{i,i-1}$ denotes the first-passage time of the birth-death process from the state $i$ to the state $i-1$
First passage time of a birth death process

- Let $\sigma_b$ denote the first-passage time to 0 of a BD process starting at $b$

\[
\sigma_b = \sigma_{b,b-1} + \sigma_{b-1,b-2} + \cdots + \sigma_{1,0}
\]

where $\sigma_{i,i-1}$ denotes the first-passage time of the birth-death process from the state $i$ to the state $i - 1$

- The Laplace transform of $\sigma_b$

\[
\hat{f}_b(s) = \left( -\frac{1}{\lambda} \right)^b \left( \prod_{i=1}^{b} \Phi_{k=i}^{\infty} \frac{-\lambda \mu_k}{\lambda + \mu_k + s} \right).
\]
Probability of the mid price moving up: spread=1

σ_b is the random time when a bid queue with b orders disappears.
σ_a is the random time when an ask queue with a orders disappears.

**Theorem**

\[ P_{a,b} \equiv \mathbb{P}[\sigma_a < \sigma_b] \]

is given by the inverse Laplace transform of

\[ \hat{F}_{a,b}(s) = \frac{1}{s} \hat{f}_b(s)\hat{f}_a(-s), \]

evaluated at \( t = 0 \), where

\[ \hat{f}_b(s) = \left(-\frac{1}{\lambda}\right)^b \left(\prod_{i=1}^b \Phi_{k=i}^{\infty} \frac{-\lambda(\mu + k\theta)}{\lambda + (\mu + k\theta) + s}\right). \]
Theorem
For $S(0) = p_A(0) - p_B(0) > 1$, the quantity

$$\mathbb{P}[^\sigma_A \wedge \sigma^\Sigma_B < \sigma_B \wedge \sigma^\Sigma_A]$$

is given by the inverse Laplace transform of

$$\hat{F}_{a,b,S}(s) = \frac{1}{s} \hat{h}_b(s)\hat{h}_a(-s),$$

evaluated at $t = 0$, where

$$\hat{h}_b(s) = \left( \hat{f}_b(s + \Lambda) + \frac{\Lambda(\hat{f}_b(s + \Lambda) - 1)}{\Lambda + s} \right)$$

and

$$\Lambda \equiv \sum_{i=1}^{S-1} \lambda(i).$$
Probability of executing an order before the price moves

\( \epsilon_b \) is the random time when the \( b \)th order at the bid is executed, given that it is not cancelled.

Theorem

\[
P_{a,b} \equiv \mathbb{P}[\epsilon_b < \sigma_a]
\]

is given by the inverse Laplace transform of

\[
\hat{F}_{a,b}(s) = \frac{1}{s} \hat{g}_b(s) \hat{f}_a(-s),
\]

evaluated at \( t = 0 \), where

\[
\hat{g}_b(s) = \prod_{i=1}^{b-1} \frac{\mu + i\theta}{\mu + i\theta + s}.
\]
Proposition

The probability

\[ P_{a,b} \equiv \mathbb{P}[\max\{\epsilon_b, \epsilon_a\} < \min\{\sigma_b, \sigma_a\}] \]

of making the spread is given by \( h_{a,b} + h_{b,a} \), where

\[
h_{a,b} = \sum_{i=0}^{\infty} \sum_{j=1}^{a} \mathbb{P}[\epsilon_j < \sigma_i] \int_0^{\infty} P_{0,i}^X(t)P_{a,j}^W(t)g_b(t)dt,
\]

where \( g_b \) is the inverse Laplace transform of \( \hat{g}_b \) and...
Probability of making the spread

**Proposition**

\[ P_{0,i}(t) = \frac{e^{-\lambda^X(t)} \lambda^X(t)^i}{i!}, \quad \lambda^X(t) = \frac{\lambda}{\theta} \left(1 - e^{-\theta t}\right) \]

\[ P_{a,j}(t) \equiv \left(e^{Q^W_a t}\right)_{a,j} \equiv \left(\sum_{k=0}^{\infty} \frac{t^k}{k!} (Q^W_a)^k\right)_{a,j} \]

\[ Q^W_a \equiv \begin{bmatrix}
0 & 0 & 0 & \cdots & 0 \\
\mu & -\mu & 0 & \cdots & 0 \\
0 & \mu + \theta & -\mu - \theta & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & \mu + (a-1)\theta & -\mu - (a-1)\theta
\end{bmatrix}. \]
## Probability of increase in mid price

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**Table:** Empirical frequencies (top) and Laplace transform results (bottom).
Probability of executing a limit order

- My order is the $b$th order at the bid.
- The ask has $a$ orders.
- The probability that my order is executed before the ask moves:

\[
\begin{array}{cccccc}
\hline
b & 1 & 2 & 3 & 4 & 5 \\
\hline
1 & .503 & .698 & .794 & .848 & .882 \\
2 & .359 & .551 & .664 & .737 & .787 \\
3 & .291 & .465 & .578 & .656 & .713 \\
4 & .251 & .409 & .517 & .596 & .654 \\
5 & .224 & .369 & .472 & .548 & .607 \\
\hline
\end{array}
\]
Probability of making the spread

- I have one limit order that is bth order at the bid.
- I have one limit order that is ath order at the ask.
- The probability that both are executed before the mid price moves:

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Conclusion

- We propose to model an order book as a continuous-time Markov chain
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- Conditional on the best bid and ask prices, each price level is a standard \((M/M/1+M)\) queue.
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- Conditional on the best bid and ask prices, each price level is a standard (M/M/1+M) queue
- We estimate model parameters with ”Level II” trades and quotes data
Conclusion

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• We find that our simulation is comparable to the data
We propose to model an order book as a continuous-time Markov chain. Conditional on the best bid and ask prices, each price level is a standard (M/M/1+M) queue. We estimate model parameters with “Level II” trades and quotes data. We find that our simulation is comparable to the data. We use Laplace transform methods to compute conditional probabilities.
Conclusion

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- Thanks! Any questions?