Mathematical Finance and Partial Differential Equations

Paul Feehan

April 20, 2009
A Brief Introduction to Financial Derivatives and Options
Overview
Options and Derivatives in Financial Markets

Introduction to Mathematical Finance
What is Brownian Motion?
Stochastic Processes
Stochastic Calculus
Mathematical Option Pricing Theory

Beyond the Black-Scholes-Merton Model and Current Research
Heston and Bates Models and Degenerate Parabolic PIDEs
Gyöngy’s Theorem and Dupire’s Local Volatility Model
Numerical Analysis and Applications to Option Pricing

References
The Nature of Financial Derivatives

An option is a contract between a buyer and a seller that gives the buyer the right but not the obligation to buy or to sell a particular asset at a later date for an agreed price.
The Nature of Financial Derivatives

- An option is a contract between a buyer and a seller that gives the buyer the right but not the obligation to buy or to sell a particular asset at a later date for an agreed price.
- In return for granting the option, the seller collects a payment (the premium) from the buyer.
The Nature of Financial Derivatives

- An option is a contract between a buyer and a seller that gives the buyer the right but not the obligation to buy or to sell a particular asset at a later date for an agreed price.
- In return for granting the option, the seller collects a payment (the premium) from the buyer.
- If the buyer chooses to exercise this right, the seller is obliged to sell or buy the asset at the agreed price; the buyer may choose not to exercise the right.
The Nature of Financial Derivatives

▶ An option is a contract between a buyer and a seller that gives the buyer the right but not the obligation to buy or to sell a particular asset at a later date for an agreed price.
▶ In return for granting the option, the seller collects a payment (the premium) from the buyer.
▶ If the buyer chooses to exercise this right, the seller is obliged to sell or buy the asset at the agreed price; the buyer may choose not to exercise the right.
▶ The underlying asset can be a piece of property, or shares of stock or some other security.
The Nature of Financial Derivatives

- An option is a contract between a buyer and a seller that gives the buyer the right but not the obligation to buy or to sell a particular asset at a later date for an agreed price.
- In return for granting the option, the seller collects a payment (the premium) from the buyer.
- If the buyer chooses to exercise this right, the seller is obliged to sell or buy the asset at the agreed price; the buyer may choose not to exercise the right.
- The underlying asset can be a piece of property, or shares of stock or some other security.
- Upon the option holder’s choice to exercise the option, the party who sold the option must fulfill the contract.
Determining the *fair value* of the option premium and how to *hedge the risks* in an option was a major unsolved problem prior to the work of Black, Scholes, and Merton (1973) and continues to be an active area of research among academic scientists and industry practitioners.
Valuation of Financial Derivatives

- Determining the *fair value* of the option premium and how to *hedge the risks* in an option was a major unsolved problem prior to the work of Black, Scholes, and Merton (1973) and continues to be an active area of research among academic scientists and industry practitioners.

- The growth of option contract complexity, sophistication stochastic process models for asset prices, and extension to new markets (equities, interest rates, commodities, for exchange, loans and bonds, credit contracts to protect against default, real estate, environment, etc) has led to an enormous growth in research in mathematical finance by mathematicians, computer scientists, statisticians, economists.
Examples of Option Contract Payoffs

Suppose \( \{S(t)\}_{t \in [0, T]} \) is a stochastic process representing an asset price (for example, a stock or bond) and that \( M(t) = \max_{u \in [0, t]} S(u) \) is its *running maximum* process.

- European-style call option with *strike* \( K \) and *maturity* \( T \) has *payoff* \( (S(T) - K)^+ = \max\{S(T) - K, 0\} \).
- European-style put option with strike \( K \) and maturity \( T \) has *payoff* \( (K - S(T))^+ \).
- European-style call option with strike \( K \), maturity \( T \), and *upper knockout barrier* \( U \) has payoff \( (S(T) - K)^+1_{M(T) \leq U} \).
- *European*-style options can *only* be exercised at maturity, \( T \).
Examples of Option Contract Payoffs (continued ...)

- *American*-style options can be exercised at any $\tau \in [0, T]$.
- The payoffs of *Asian* options depend on the average price of an asset over time and are popular in commodities markets (for example, oil, wheat, etc).
- *Interest rate swaps* allow a client to pay for the privilege of a fixed interest rate in exchange for a variable interest rate (for example, fixed versus adjustable rate mortgages).
- *FX swaps* allow a client to pay for a fixed foreign exchange rate in exchange for a variable rate.
- *Credit default swaps* allow a client to pay to protect a loan or bond against default.
Einstein’s 1905 article, *On the Motion Required by the Molecular Kinetic Theory of Heat of Small Particles Suspended in a Stationary Liquid*, one of his four Annus Mirabilis papers published during 1905, he described a stochastic model of Brownian motion:

*In this paper it will be shown that, according to the molecular kinetic theory of heat, bodies of a microscopically visible size suspended in liquids must, as a result of thermal molecular motions, perform motions of such magnitudes that they can be easily observed with a microscope. It is possible that the motions to be discussed here are identical with so-called Brownian molecular motion ...*
Before Einstein’s paper, atoms were recognized as a useful concept, but physicists and chemists debated whether atoms were real entities.
Brownian Motion and Einstein’s 1905 Paper (continued ...)

- Before Einstein’s paper, atoms were recognized as a useful concept, but physicists and chemists debated whether atoms were real entities.

- The article also lent credence to statistical mechanics, which had been controversial at that time, as well.
Before Einstein’s paper, atoms were recognized as a useful concept, but physicists and chemists debated whether atoms were real entities.

The article also lent credence to statistical mechanics, which had been controversial at that time, as well.

Einstein’s statistical discussion of atomic behavior gave experimentalists a way to count atoms by looking through an ordinary microscope.
Before Einstein’s paper, atoms were recognized as a useful concept, but physicists and chemists debated whether atoms were real entities.

The article also lent credence to statistical mechanics, which had been controversial at that time, as well.

Einstein’s statistical discussion of atomic behavior gave experimentalists a way to count atoms by looking through an ordinary microscope.

Wilhelm Ostwald, one of the leaders of the anti-atom school, was convinced of the existence of atoms by Einstein’s complete explanation of Brownian motion.
Brownian Motion and Louis Bachelier’s 1900 Thesis

Louis Jean-Baptiste Alphonse Bachelier (1870–1946) was a French mathematician credited with being the first person to model Brownian motion, which was part of his Ph.D. thesis, *The Theory of Speculation* (1900), supervised by Henri Poincaré.

- Bachelier’s paper was the first to give mathematical developments of Brownian motion and option pricing.
Brownian Motion and Louis Bachelier’s 1900 Thesis

Louis Jean-Baptiste Alphonse Bachelier (1870–1946) was a French mathematician credited with being the first person to model Brownian motion, which was part of his Ph.D. thesis, *The Theory of Speculation* (1900), supervised by Henri Poincaré.

- Bachelier’s paper was the first to give mathematical developments of Brownian motion and option pricing.
- Today, he is considered a pioneer in the study of mathematical finance and stochastic processes.
Brownian Motion and Louis Bachelier’s 1900 Thesis

Louis Jean-Baptiste Alphonse Bachelier (1870–1946) was a French mathematician credited with being the first person to model Brownian motion, which was part of his Ph.D. thesis, *The Theory of Speculation* (1900), supervised by Henri Poincaré.

- Bachelier’s paper was the first to give mathematical developments of Brownian motion and option pricing.
- Today, he is considered a pioneer in the study of mathematical finance and stochastic processes.
- Despite the now undisputed brilliance of his doctoral thesis, Bachelier spent the rest of his academic career languishing in a provincial university in the French countryside.
Wiener’s Definition of Brownian Motion

- Observed by Robert Brown as a physical phenomenon in 1828.
- Treated mathematically by Bachelier in 1900, Einstein in 1905.
- First mathematically rigorous construction of Brownian motion is due to Wiener (1923, 1924).
- Further non-intuitive properties of Brownian motion discovered by Paul Lévy (1939, 1948).
Definition

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space. For each \(\omega \in \Omega\), suppose there is a continuous function \(W(t) = W(\omega, t)\) of \(t \geq 0\) with \(W(0) = 0\) and which depends on \(\omega\). Then \(\{W(t)\}_{t \geq 0}\) is Brownian motion if for all \(0 = t_0 < t_1 < \cdots < t_m\), the increments

\[W(t_1) - W(t_0), W(t_2) - W(t_1), \ldots, W(t_m) - W(t_{m-1})\]

are independent and each increment is normally distributed with

\[\mathbb{E}_{\mathbb{P}}[W(t_{i+1}) - W(t_i)] = 0,\]
\[\mathbb{E}_{\mathbb{P}}[(W(t_{i+1}) - W(t_i))^2] = t_i - t_{i-1}.\]
Alternative Characterization of Brownian Motion

Definition
Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space. For each \(\omega \in \Omega\), suppose there is a continuous function \(W(t) = W(\omega, t)\) of \(t \geq 0\) with \(W(0) = 0\) and which depends on \(\omega\). Then \(\{W(t)\}_{t \geq 0}\) is Brownian motion if for all \(0 = t_0 < t_1 \cdots < t_m\), the random variables \(W(t_1), W(t_2), \ldots, W(t_m)\) are jointly normally distributed with zero mean and covariance \(\mathbb{E}_{\mathbb{P}}[(W(t_{i+1}) - W(t_i))^2] = \min\{t_i, t_{i-1}\}\).
Definition
Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space and let \(T > 0\). A collection of random variables, \(\{X(t)\}_{t \in [0,T]}\), is called a stochastic process. If \(\mathbb{F} = \{F(t)\}_{t \geq 0}\) is a filtration for \(\mathcal{F}\), then \(\{X(t)\}_{t \in [0,T]}\) is an adapted stochastic process if, for each \(t \in [0, T]\), the random variable \(X(t)\) is \(\mathcal{F}(t)\)-measurable.

- A random variable, \(Y\), is an \(\mathcal{F}\)-measurable function, \(Y : \Omega \rightarrow \mathbb{R}\).
- Brownian motion, \(\{W(t)\}_{t \in [0,T]}\), is a stochastic process.
Filtrations for Brownian Motion

We will need a concept of “information flow” and the gradual revelation of information in financial markets:

**Definition**

Let \( \{ W(t) \}_{t \geq 0} \) be a Brownian motion on \((\Omega, \mathcal{F}, P)\). A *filtration* for \( \{ W(t) \}_{t \geq 0} \) is a collection of \( \sigma \)-algebras \( \mathbb{F} = \{ F(t) \}_{t \geq 0} \) such that

- For \( 0 \leq s < t \), \( F(s) \subset F(t) \subset \mathcal{F} \),
- \( W(t) \) is \( F(t) \)-measurable,
- For \( 0 \leq t < u \), \( W(t) - W(u) \) is independent of \( F(t) \).
Martingales

The concept of a \textit{martingale} is essential in mathematical finance:

\textbf{Definition}

Let $\Omega, F, P$ be a probability space, let $T > 0$, and let

$\{X(t)\}_{t \in [0, T]}$ be an adapted stochastic process. Then $\{X(t)\}_{t \geq 0}$

is a \textit{martingale} if for all $0 \leq s \leq t \leq T$,

$$
E_P[X(t) | F(s)] = X(s).
$$

Informally, the process $X(t)$ has no tendency to either rise or fall. Brownian motion is a martingale.
Definition

Let \( f : [0, T] \to \mathbb{C} \) be a function. The quadratic variation of \( f \) up to time \( t \in [0, T] \) is

\[
[f, f](t) = \lim_{\|\Pi\| \to 0} \sum_{j=0}^{n-1} |f(t_{j+1}) - f(t_j)|^2,
\]

where \( \Pi = \{t_0, t_1, \ldots, t_n\} \) and \( 0 = t_0 < t_1 < \cdots < t_n = t \).

- If \( f \in C^1(\mathbb{R}) \), one can show that \( [f, f](t) = 0 \).
- We often write \( df \cdot df \) or \((df)^2\) instead of \([f, f]\).
Lévy’s characterization of Brownian motion

**Theorem**

Let \( \{ W(t) \}_{t \geq 0} \) be a Brownian motion on \( (\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F}) \). Then \( [W, W](t) = t \) for all \( t \geq 0 \) almost surely.

Brownian motion paths \( W(\omega, t) \) are nowhere differentiable with respect to \( t > 0 \).

**Theorem (Lévy)**

Let \( \{ M(t) \}_{t \geq 0} \) be a martingale on \( (\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F}) \), with \( M(0) = 0 \), continuous paths \( M(\omega, \cdot) \), and quadratic variation \( [M, M](t) = t \) for all \( t \geq 0 \). Then \( \{ M(t) \}_{t \geq 0} \) is a Brownian motion.
Stochastic Integrals

Definition
Let \( \{ W(t) \}_{t \geq 0} \) be a Brownian motion on \( (\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}) \). Let \( T > 0 \) and let \( \Pi = \{ t_0, t_1, \ldots, t_n \} \) be a partition of \([0, T]\). Let \( \Delta(u) \) be a simple process, so \( \Delta(u) \) is constant on each subinterval \([t_j, t_{j+1})\).

For \( t \in [t_k, t_{k+1}] \), the Itô integral of \( \Delta(u) \) with respect to \( W(u) \) is

\[
\int_0^t \Delta(u) \, dW(u) := \sum_{j=0}^{k-1} \Delta(t_j)[W(t_{j+1}) - W(t_j)] + \Delta(t_k)[W(t) - W(t_k)].
\]

If \( \Delta(u) \) is any adapted stochastic process (with mild technical conditions), the integral is defined as a limit.
Itô Processes

Definition
An Itô process \( \{X(t)\}_{t \geq 0} \) is a stochastic process of the form

\[
X(t) = X(0) + \int_0^t \theta(u) \, du + \int_0^t \sigma(u) \, dW(u),
\]

where \( X(0) \) is non-random and \( \theta(u), \sigma(u) \) are adapted stochastic processes. We often abbreviate this “integral equation” as

\[
dX(t) = \theta(t) \, dt + \sigma(t) \, dW(t), \quad X(0) = X_0.
\]

One can show that \([X, X](t) = \int_0^t \sigma^2(u) \, du\).
Theorem

Let \( \{X(t)\}_{t \geq 0} \) be an Itô process and let \( f : [0, \infty) \times \mathbb{R} \to \mathbb{R} \) be a \( C^{1,2} \) function. Then, for all \( t \geq 0 \),

\[
f(t, X(t)) = f(0, X(0)) + \int_0^t f_u(u, X(u)) \, du + \int_0^t f_x(u, X(u)) \, dX(u) \\
+ \int_0^t f_{xx}(u, X(u)) \, d[X, X](u),
\]

Informally, one writes

\[
\text{df}(t, X(t)) = f_t(t, X(t)) \, dt + f_x(t, X(t)) \, dX(t) + f_{xx}(t, X(t)) \, d[X, X](t).
\]
New Martingales from Old and the Martingale Representation Theorem

**Theorem**

Let \( \{ W(t) \} \) be a Brownian motion on \((\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})\), where \( \mathcal{F}(t) \) is generated by \( W(t) \). If \( M(t) \) is a martingale, then there is an adapted process \( \{ \Gamma(t) \} \) such that

\[
M(t) = M(0) + \int_0^t \Gamma(u) \, dW(u), \quad 0 \leq t \leq T.
\]

Informally, \( M(t) \) is a martingale if there is no drift or “\( dt \)” term in \( dM(t) \):

\[
dM(t) = \Gamma(t) \, dW(t).
\]
Examples of Stochastic Processes Driven by Brownian Motion

- Geometric Brownian motion was used as a stock price model used by Black, Scholes, and Merton (1973):

\[ dS(t) = S(t) (\mu \, dt + \sigma \, dW(t)) . \]

We call \( \mu \) the \textit{drift} and \( \sigma \) the \textit{volatility} of the stock.

- The Cox-Ingersoll-Ross (or Feller or square-root) process is a simple model for interest rate processes:

\[ dR(t) = \kappa (\theta - R(t)) \, dt + \xi \sqrt{R(t)} \, dW(t) \]
Stochastic Differential Equations and their Solutions

Let $\beta(u, x)$, $\gamma(u, x)$ be deterministic functions. Then

$$dX(u) = \beta(u, X(u)) \, du + \gamma(u, X(u)) \, dW(u), \quad X(t) = x,$$

is a stochastic differential equation (SDE) for a process $\{X(u)\}_{u \geq t}$. Using $X(t) = \log S(t)$, where $S(t)$ is geometric Brownian motion, the Itô-Doeblin gives

$$dX(t) = (\mu - \sigma^2/2) \, dt + \sigma \, dW(t),$$

and hence

$$S(t) = S(0) \exp \left((\mu - \sigma^2/2)t + \sigma W(t)\right) = f(t, W(t)), \quad t > 0.$$
Markov Processes

**Definition**

Let $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ be a filtered probability space and let 
$
\{X(t)\}_{t \in [0, T]} 
$
be an adapted stochastic process. Suppose that for every Borel-measurable function $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ one has

$$
\mathbb{E}_{\mathbb{P}}[f(t, X(t)) | \mathcal{F}(s)] = f(s, X(s)), \quad 0 \leq s \leq t \leq T,
$$

then $X(t)$ is a *Markov process*.
Examples of Markov and Martingale Processes

- Brownian motion, geometric Brownian motion, and the Cox-Ingersoll-Ross processes are Markov.
- If $S(t)$ is geometric Brownian motion and $\mu = r$, the risk free interest rate, then the Itô-Doeblin formula shows that $e^{-rt}S(t)$ is a martingale.
- Arithmetic Brownian motion, $X(t) = at + bW(t)$, is Markov but not a martingale unless $a = 0$.
- Martingales need not be Markov.
Let $h : \mathbb{R} \to \mathbb{R}$ be a Borel-measurable function and denote
\[
g(t, x) = \mathbb{E}_P^{t,x} [h(X(T))],
\]
where $X(T)$ is the SDE solution with initial condition $X(t) = x$.

**Theorem**

Let $\{X(u)\}_{u \in [0, T]}$ be a solution to an SDE with initial condition $X(0) = X_0$. Then
\[
\mathbb{E}_P[h(X(T))|\mathcal{F}(t)] = g(t, X(t)),
\]
and $\{X(u)\}_{u \in [0, T]}$ is Markov.
Connections between Stochastic Differential Equations and Partial Differential Equations

**Theorem (Feynman-Kac)**

Let \( \{X(u)\}_{u \in [0,T]} \) be a solution to

\[
dX(u) = \beta(u, X(u)) \, du + \gamma(u, X(u)) \, dW(u), \quad X(t) = x,
\]

and let \( h \) be Borel-measurable. Then \( g(t, x) = \mathbb{E}_{P}^{t,x}[h(X(T))] \) obeys

\[
g_t(t, x) + \beta(t, x)g_x(t, x) + \frac{1}{2}\gamma^2(t, x)g_{xx}(t, x) = 0, \quad (t, x) \in [0, T) \times \mathbb{R},
\]

with terminal condition \( g(T, x) = h(x) \) for all \( x \in \mathbb{R} \).
Martingales and Risk-Neutral Pricing Formula

Suppose the payoff of an option contract is represented by an $\mathcal{F}(T)$-measurable random variable, $H$, on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{Q}, \mathbb{F})$. Suppose our financial market does not permit arbitrage, $V(t)$ is the fair (or no-arbitrage) price of the option contract at time $t \in [0, T]$ (in particular, $V(T) = H$), and that the risk free interest rate is $r \geq 0$. Then $e^{-rt}V(t)$ is a martingale and

$$e^{-rt}V(t) = \mathbb{E}_\mathbb{Q}[e^{-rT}V(T) | \mathcal{F}(t)].$$
Fischer Black, Myron Scholes, Robert Merton and their 1973 Nobel-prize Winning Discovery

**Theorem**

*Let the price of an asset, $S(t)$, be represented by geometric Brownian motion, $dS(t) = S(t)(\mu \, dt + \sigma \, dW(t))$. Then the fair price, $V(t)$, at time $t \in [0, T]$ of a call option with strike price $K > 0$ and maturity $T > 0$ is given by*

$$V(t) = S(t)N(d_+(T - t, S(t))) - e^{r(T-t)}KN(d_-(T - t, S(t))),$$

*where*

$$d_{\pm}(\tau, x) = \frac{1}{\sigma \sqrt{\tau}} \left[ \log \left( \frac{x}{K} \right) + \left( r - \frac{\sigma^2}{2} \right) \tau \right].$$
How to Derive the Black-Scholes-Merton Formula

There are two theoretical approaches:

- Use probability theory to compute the expectation

\[ V(t) = e^{-r(T-ty)} \mathbb{E}_Q[(S(T) - K)^+ | \mathcal{F}(t)] \].

- Or observe that \( V(t) = g(t, S(t)) \) for some deterministic function \( g(t, x) \) and solve the Black-Scholes-Merton PDE:

\[
g_t(t, x) + rxg_x(t, x) + \frac{1}{2} \sigma^2 x^2 g_{xx}(t, x) = rg(t, x), \quad (t, x) \in [0, T) \times \mathbb{R},
\]

with terminal condition \( g(T, x) = (x - K)^+ \) for all \( x \geq 0 \).
Deficiencies in the Black-Scholes-Merton Model

- Empirical observation tells us that asset prices are not accurately modeled by geometric Brownian motion.
- Use of such simple models lead to serious mispricing and poor risk management.
- Since 1987, researchers have sought to improve and extend the Black-Scholes-Merton Model.
Heston’s Stochastic Volatility Model

Heston’s asset price process, $S(u)$, is defined by

$$
\begin{align*}
    dS(u) &= S(u) \left( \alpha(u) \, du + \sqrt{V(u)} \, dW_1(u) \right), \quad S(t) = s, \\
    dV(u) &= (a - bV(u)) \, du + \xi \sqrt{V(u)} \, dW_2(u), \quad V(t) = y,
\end{align*}
$$

where $(W_1(u), dW_3(u))$ is two-dimensional Brownian motion, $W_2(u) := \rho W_1(u) + \sqrt{1 - \rho^2} \, W_3(u)$, $a, b, \xi, \rho$ are positive constants, $\alpha(u)$ is an adapted process, and $V(u)$ is the variance process.
Bate’s Stochastic Volatility with Jumps Model

Bate’s asset price process, $S(t)$, can be defined by

$$dS(t) = S(t-) \left( \alpha dt + \sqrt{V(t)} \, dW_1(t) + dQ^\ell(t) \right), \quad S(0) = S_0,$$

$$dV(t) = (a - bV(t)) \, dt + \xi \sqrt{V(t)} \, dW_2(t), \quad V(0) = V_0,$$

where $\alpha$ and $\rho$ are constants, and $S_0$, $V_0$, $\xi$, $a$, and $b$ are positive constants, $(W_1(t), dW_3(t))$ is two-dimensional Brownian motion, $W_2(t) := \rho W_1(t) + \sqrt{1 - \rho^2} \, W_3(t)$, and $Q^\ell(t) = \sum_{i=1}^{N(t)} Y_i^{\ell}$ is a compound Poisson process with intensity $\lambda > 0$ and $\{1 + Y_i^{\ell}\}_{i=1}^\infty$ is a sequence of independent, lognormal random variables with mean $1 + \mu^\ell := \mathbb{E}_P[1 + Y_i^{\ell}]$ and variance $\delta^2_{\ell} := \text{Var}[1 + Y_i^{\ell}]$ (so $Y_i^{\ell}$ has mean $\mu^\ell = \mathbb{E}_P[Y_i^{\ell}]$ and variance $\delta^2_{\ell} = \text{Var}_P[Y_i^{\ell}]$).
Degenerate Parabolic Integro-Differential Equations

The option pricing problem for the Heston process leads to a degenerate parabolic PDE. If \( g(t, s, y) := \mathbb{E}_Q^{t,s,y}[h(S(T), V(T))] \), then the function \( g(t, s, y) \) obeys

\[
g_t + rsg_s + (a - by)g_y + \frac{1}{2} \left( s^2 yg_{ss} + 2\rho\xi syg_{sy} + \xi^2 yg_{yy} \right) = 0,
\]

with final condition \( g(T, s, y) = h(s, y) \). If the Heston process is augmented by a jump-diffusion or, more generally, a Lévy process, one obtains a degenerate parabolic PIDE.
Gyöngy’s Theorem

Theorem (Gyöngy, 1986)

Let \( W \) be an \( \mathbb{R}^{n_1} \)-valued Wiener process, and let \( X \) be an \( \mathbb{R}^d \)-valued Itô process with stochastic differential

\[
dX(t) = \mu(t) \, dt + \sigma(t) \, dW(t),
\]

where \( \mu(t) \in \mathbb{R}^d \) and \( \sigma(t) \in \mathbb{R}^d \times \mathbb{R}^{n_1} \) are adapted processes satisfying

\[
\mathbb{E} \left[ \int_0^t \left( \| \mu \| + \| \sigma \sigma^T \| \right) \, ds \right] < \infty, \quad \forall t \in \mathbb{R}^+.
\]
Gyöngy’s Theorem (continued ...)

Theorem (continued ...)

Let $\hat{\mu} : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}^d$ and $\hat{\sigma} : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}^d \times \mathbb{R}^{n_2}$ be (deterministic) functions such that $\hat{\mu}(X(t), t) = \mathbb{E}[\mu(t)|X(t)]$ and $\hat{\sigma}(X(t), t) = \mathbb{E}[\sigma(t)\sigma^T(t)|X(t)]$. Then there exists a weak solution to the SDE

$$d\hat{X}(t) = \hat{\mu}(t)(\hat{X}(t), t) \, dt + \hat{\sigma}(\hat{X}(t), t) \, d\hat{W}(t),$$

where $\hat{W}(t)$ is an $\mathbb{R}^{n_2}$-valued Wiener process, and $\hat{X}(t)$ has the same one-dimensional marginal distributions as $X(t)$. 
Suppose the market price, \( C(t, T, x, K) \), at time \( t \) of a European-style call option with strike \( K \), maturity \( T \), and current asset price \( S(t) = x \) is known for all \( K > 0 \) and \( T > 0 \). Dupire showed that if a stock price process, \( S(u), u \geq t \), is modeled using

\[
dS(u) = S(u) \left( r \, dt + \sigma_{t,x}(u, S(u)) \, dW(u) \right), \quad S(t) = x,
\]

where \( \sigma_{t,x}(\cdot, \cdot) \) is a deterministic function defined by

\[
\sigma_{t,x}^2(T, K) = \frac{C_T(t, T, x, K) + rKC_K(t, T, x, K)}{\frac{1}{2}K^2C_{KK}(t, T, x, K)},
\]

then model prices will exactly match market prices for all \( K > 0 \) and \( T > 0 \), for fixed \((t, x)\).
Gyöngy’s Theorem and the Dupire Model

- Dupire’s formula for $\sigma_{t,x}(\cdot,\cdot)$ implements Gyöngy’s Theorem to solve option pricing problems.
- Dupire’s model accurately prices simple European-style call or put options: the local volatility process has the same one-dimensional marginal distributions as the “market” asset price process (which may be much more complex).
- Dupire’s model will misprice options requiring exact knowledge of higher-dimensional distributions. For example, to exactly price a European-style “up-and-out” barrier option one requires the joint probability distribution of the stock price process, $S(t)$, and its running maximum, $M(t)$. 
Traders usually pretend that the \((t, x)\) used when defining \(\sigma_{t,x}(\cdot, \cdot)\) remains fixed, but this leads to mispricing.

Traders make ad hoc corrections to Dupire’s model.

More accurate models would allow \(\sigma_{t,S(t)}(\cdot, \cdot)\) to evolve as a 2-dimensional stochastic surface.

**Stochastic surface models** are infinite-dimensional and lead to deep, largely unsolved problems in partial differential equations, infinite-dimensional stochastic analysis, and numerical implementation.

Practical implementations of a stochastic surface model would seek to replace it by a finite-dimensional, Markov model.
Brunick’s Extension of Gyöngy’s Theorem

- Gerard Brunick’s dissertation (2008) significantly extended Gyöngy’s Theorem by proving that it was possible to create a new simpler process which replicated higher-order distributions for a given complex process.

- Brunick’s Theorem and possible extensions should have significant applications to pricing and hedging problems for exotic or complex (path-dependent) options.
Numerical Solution of Option Pricing Problems

- Monte Carlo simulation,
- Trees and lattices,
- Characteristic functions and solutions via the fast Fourier transform (FFT),
- Closed-form solution of PIDEs,
- Finite difference solution of PIDEs,
- Finite element solution of PIDEs.
References