Geometry of functionally generated portfolios

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Multiplicative Cyclical Monotonicity

Portfolio as a function on the unit simplex

- Δ - unit simplex in dimension n
- Market weights for n stocks:
- μ_i = Proportion of the total capital that belongs to ith stock.
- Process in time, $\mu(t),\ t = 0, 1, 2, \ldots$ in Δ.
- Portfolio: $\pi = (\pi_1, \ldots, \pi_n) \in \Delta$. Process in time $\pi(t)$.
- Portfolio weights:

$$\pi_i = \text{Proportion of the total value that belongs to } i\text{th stock.}$$

- For us $\pi = \pi(\mu) : \Delta \to \overline{\Delta}$.

Relative value

- How does the portfolio π compare with an index, say, S&P 500?
- Start by investing \$1 in portfolio and compare with index.
- Relative value process: $V(\cdot)$ = ratio of growth of \$1.

$$\frac{\Delta V(t)}{V(t)} = \sum_{i=1}^{n} \pi_i(t) \frac{\Delta \mu_i(t)}{\mu_i(t)}, \quad V(0) = 1.$$

- Δ^* - subset of unit simplex (e.g. simplex with cut corners).
- $\pi = \pi(\mu)$ pseudo-arbitrage on Δ^* if $\exists\ \epsilon > 0$, $V(t) \geq \epsilon$ for all possible paths $\{\mu(\cdot)\} \subseteq \Delta^*$. $\lim_{t\to\infty} V(t) = \infty$ for some path.

Does there exist pseudo-arbitrage portfolio functions?

- The special case of cycles.

$$\begin{tikzcd}
& \mu_1 \arrow[dr] & \\
\mu_0 \arrow[ur] & & \mu_2 \arrow[d] \\
\mu_m \arrow[u] & & \mu_3 \arrow[dl] \\
& \mu_4 \arrow[ul] &
\end{tikzcd}$$

- Market cycles through a sequence of size m.
- Let $\eta = V(m+1)$. Dichotomy: $\eta < 1$, or $\eta \geq 1$.
- After k cycles: $V(k(m+1)) = \eta^k \to 0$, if $\eta < 1$.
- If π has to be a pseudo-arbitrage, it must be multiplicative cyclically monotone.

Functionally generated portfolios. Fernholz '99

Theorem (Fernholz '99, '02, P.-Wong '14)

π is MCM iff $\exists\, \Phi : \Delta^* \to (0, \infty)$, ***concave****: $\pi_i/\mu_i \in \partial \log \Phi(\mu)$. Or,*

$$\pi_i(\mu) = \mu_i \left[1 + D_{e(i)-\mu} \log \Phi(\mu)\right]$$

If Φ not affine, π is a pseudo-arbitrage in discrete/continuous time.

Outperformance over cycles $\Leftrightarrow$ asymptotic outperformance over all paths.

Examples

- $\varphi : \Delta \to \mathbb{R} \cup \{-\infty\}$ is exponentially concave if $\Phi = e^{\varphi}$ is concave.

$$\mathrm{Hess}(\varphi) + \nabla\varphi\,(\nabla\varphi)' \leq 0.$$

- Examples: $p, \pi \in \Delta$, $0 < \lambda < 1$.

$$\begin{aligned}
\varphi(\mu) &= \frac{1}{n}\sum_i \log \mu_i, \quad \pi(\mu) = (1/n, 1/n, \ldots, 1/n). \\
\varphi(\mu) &= 2\log\left(\sum_i \sqrt{\mu_i}\right), \quad \pi_i(\mu) = \frac{\sqrt{\mu_i}}{\sum_{j=1}^n \sqrt{\mu_j}}.
\end{aligned}$$

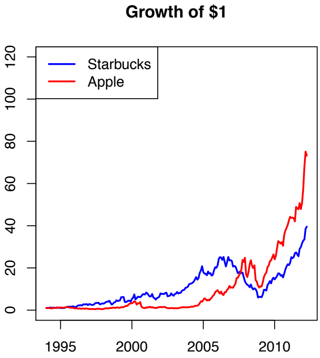
Several recent occurrences

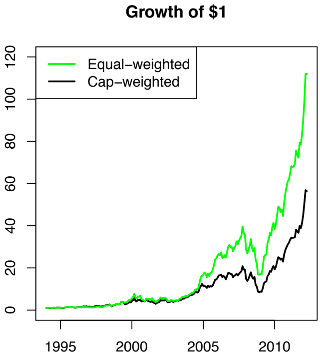
- Stochastic portfolio theory. Fernholz, Karatzas, Kardaras, Ichiba, Ruf '05 -'16.
- Entropic Curvature-Dimension conditions and Bochner's inequality. Erbar, Kuwada, and Sturm '15.
- Statistics, optimization, machine learning. Cesa-Bianchi and Lugosi '06, Mahdavi, Zhang, and Jin '15.
- Unified study is lacking. Compare log-concave functions.

The blessings of dimensionality

Apple-Starbucks example

- Pair trading: $n = 2$. $\pi \equiv (1/2, 1/2)$. Cap-weighted vs. equal weighted.





- Pair trading is risky and statistically tricky.

Concentration of measure

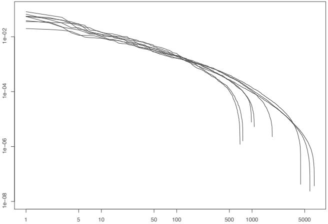
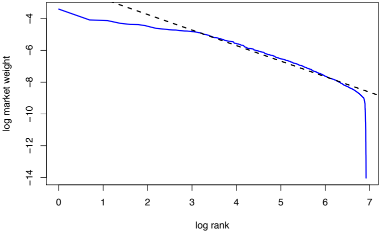


Figure 1: Capital distribution curves: 1929–1999



- Pick Δ^* by choosing a feature that is highly concentrated.
- Ordered market weights are typically Pareto: $\log \mu_{(i)} \propto i^{-\alpha}$.
- Slope $\alpha \approx 0.8$. Axtell '01 *Science*.

The Pareto distribution

- Fix $\alpha \in (1/2, 1)$. Define $\nu^{(n)} \in \Delta$ by

$$\nu_i^{(n)} = \frac{i^{-\alpha}}{\sum_{j=1}^{n} j^{-\alpha}}.$$

- Consider Dirichlet distribution $\mathrm{Dir}\left(n\nu^{(n)}\right)$.
- Assumption 1: $\left\|\mu(0) - \nu^{(n)}\right\|$ has the same distribution as $\mu(0) \sim \mathrm{Dir}(n\nu^{(n)})$.
- Assumption 2: μ is a continuous semimartingale process that is "slow to escape $O(1/\sqrt{n})$ neighborhoods of $\nu^{(n)}$".

Cosine portfolios in high dimensions

- Define exp-concave function on $\left\| \mu - \nu^{(n)} \right\| < \frac{\pi}{2\sqrt{n}}$.

$$\varphi(\mu) = \log\cos\left(\sqrt{n}\left\|\mu - \nu^{(n)}\right\|\right).$$

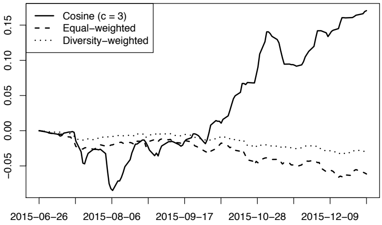
- Concentration: Under $\mathrm{Dir}\left(n\nu^{(n)}\right)$,

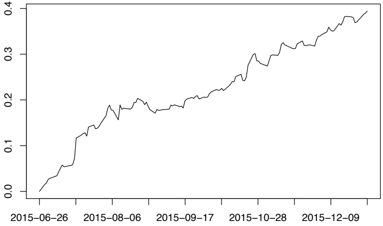
$$P\left(\mu : \left\|\mu - \nu^{(n)}\right\| < \frac{\pi}{2\sqrt{n}}\right) \approx 1.$$

- (P. '16) $\exists\, g_n = O(n^{\alpha - 1/2})$, $1/2 < \alpha < 1$, such that

$$P\left(\log V(1/\sqrt{\log n}) \geq g_n\right) = 1 - O\left(\exp\left(-c_0 n^{(1-\alpha)/4}\right)\right).$$

Performance of cosine portfolios





- $n = 1000$. $\alpha \in [0.75, 0.95]$. Jun - Dec 2015.
- Distance from Pareto scales like $\sqrt{n}$.
- Beats the index by 15% in 6 months.

What is the optimal frequency of rebalancing?

Main question

- What is the optimal frequency of rebalancing?
- Weekly/ monthly/ daily/ every second ?
- Suppose $\mu(0) = p$, $\mu(1) = q$, $\mu(2) = r$.
- I can rebalance at (i) $t = 0, 1, 2$ or at (ii) $t = 0, 2$.
- Problem: Given φ exp-concave, can I characterize $(p, q, r) \in \Delta^3$ such that (ii) is better than (i).
- I.e., when is trading less frequently better?

A new information geometry

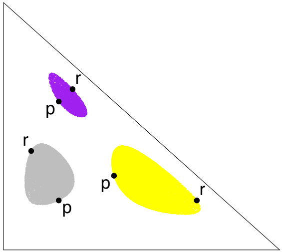


Figure : Plots of q when less trading is better

- "Theorem". (P. and Wong '16) Take any q on boundary. Then (p, q, r) forms a "right angle triangle". The sides are geodesics of a geometry and angles are given by a Riemannian metric.

Monge-Kantorovich transport problem

- P, Q - probability measures on Polish spaces $\mathcal{X}, \mathcal{Y}$.
- $c : \mathcal{X} \times \mathcal{Y} \to [-\infty, \infty]$ - cost function.
- Π - set of couplings of P, Q. Probabilities on $\mathcal{X} \times \mathcal{Y}$.

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- Π - set of couplings of P, Q. Probabilities on $\mathcal{X} \times \mathcal{Y}$.

- Find solution to

$$\inf_{R \in \Pi} R\left(c(X, Y)\right).$$

- If inf is finite, call value. Solution R - optimal coupling.

Cost - log moment generating function

- $\mathcal{X} = \overline{\Delta}$, $\mathcal{Y} = [-\infty, \infty)^n$.

$$c(\mu, \theta) = \log \sum_{i=1}^{n} e^{\theta_i} \mu_i = \log \mu \left(e^{\theta} \right).$$

- Consider

$$\inf R\left(c(\mu, \theta)\right), \quad \text{over all couplings of } (P, Q).$$

- Solution is an optimal coupling (μ, θ).

Exponential change of measures

Theorem (P.-Wong '14)

Consider optimal coupling (μ, θ) *for some* (P, Q)*. Let*

$$\pi_i = \frac{e^{\theta_i}\mu_i}{\sum_j e^{\theta_j}\mu_j}, \quad i = 1, \ldots, n.$$

Then $\pi = \pi(\mu)$ *is a Pseudo-arbitrage on appropriate* Δ^**.*

Conversely every pseudo-arbitrage can be obtained as an optimal coupling for this cost function.

The "geometry" is given by this transport.

Thank you

For more details, see:

- arxiv.org/abs/1402.3720
- arxiv.org/abs/1605.05819
- arxiv.org/abs/1603.01865

The End. Thank you.