Valuation and Hedging of Credit Default Swaptions

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References on Valuation of Credit Default Swaptions

- D. Brigo and A. Alfonsi: Credit default swaps calibration and option pricing with the SSRD stochastic intensity and interest-rate model. *Finance and Stochastics* 9 (2005), 29-42.
References on Hedging of Credit Default Swaptions


- T. Bielecki, M. Jeanblanc and M. Rutkowski: Valuation and hedging of credit default swaptions in the CIR default intensity model. Forthcoming in *Finance & Stochastics*. 
Outline

1. Generic Defaultable Claims
2. Credit Default Swaps and Swaptions
3. Market Pricing Formulae
4. CIR Default Intensity Model
5. Hedging of CDS Swaptions with Forward CDS and Swap Portfolio
Basic Default Set-up: I

Terminology and notation:

1. The **default time** is a strictly positive random variable $\tau$ defined on the underlying probability space $(\Omega, \mathcal{G}, \mathbb{P})$.

2. We define the **default indicator process** $H_t = 1\{\tau \leq t\}$ and we denote by $\mathbb{H}$ its natural filtration.

3. We assume that we are given, in addition, some auxiliary filtration $\mathbb{F}$ and we write $\mathbb{G} = \mathbb{H} \vee \mathbb{F}$, meaning that $\mathcal{G}_t = \sigma(\mathcal{H}_t, \mathcal{F}_t)$ for every $t \in \mathbb{R}_+$.

4. The filtration $\mathbb{F}$ is termed the **reference filtration**.

5. The filtration $\mathbb{G}$ is called the **full filtration**.
Let us denote by

\[ G_t = \mathbb{Q}(\tau > t \mid \mathcal{F}_t) \]

the survival process of \( \tau \) with respect to the reference filtration \( \mathcal{F} \). We postulate that \( G_0 = 1 \) and \( G_t > 0 \) for every \( t \in [0, T] \).

For any \( \mathbb{Q} \)-integrable and \( \mathcal{F}_T \)-measurable random variable \( Y \), the following classic formula is valid

\[ \mathbb{E}_\mathbb{Q}(\mathbb{1}_{\{T < \tau\}} Y \mid \mathcal{G}_t) = \mathbb{1}_{\{t < \tau\}} G_t^{-1} \mathbb{E}_\mathbb{Q}(G_T Y \mid \mathcal{F}_t). \]
Defaultable Claim

A generic defaultable claim \((X, A, Z, \tau)\) consists of:

1. A promised contingent claim \(X\) representing the payoff received by the holder of the claim at time \(T\), if no default has occurred prior to or at maturity date \(T\).
2. A process \(A\) representing the dividends stream prior to default.
3. A recovery process \(Z\) representing the recovery payoff at time of default, if default occurs prior to or at maturity date \(T\).
4. A random time \(\tau\) representing the default time.

Definition

The dividend process \(D\) of a defaultable claim \((X, A, Z, \tau)\) maturing at \(T\) equals, for every \(t \in [0, T]\),

\[
D_t = X 1_{\{\tau > T\}} 1_{[T, \infty]}(t) + \int_0^t (1 - H_u) dA_u + \int_0^t Z_u dH_u.
\]
The underlying market model is arbitrage-free, in the following sense:

1. Let the savings account $B$ be given by

$$B_t = \exp \left( \int_0^t r_u \, du \right), \quad \forall \ t \in \mathbb{R}_+,$$

where the short-term rate $r$ follows an $\mathcal{F}$-adapted process.

2. A spot martingale measure $Q$ is associated with the choice of the savings account $B$ as a numéraire.

3. The underlying market model is arbitrage-free, meaning that it admits a spot martingale measure $Q$ equivalent to $\mathbb{P}$. Uniqueness of a martingale measure is not postulated.
Recall that:

- The process $B$ represents the **savings account**.
- A probability measure $\mathbb{Q}$ is a **spot martingale measure**.

**Definition**

The **ex-dividend price** $S$ associated with the dividend process $D$ equals, for every $t \in [0, T]$,

$$S_t = B_t \mathbb{E}_\mathbb{Q} \left( \int_{[t, T]} B_u^{-1} \, dD_u \mid G_t \right) = 1_{\{t < \tau\}} \tilde{S}_t$$

where $\mathbb{Q}$ is a spot martingale measure.

- The ex-dividend price represents the *(market)* **value** of a defaultable claim.
- The $\mathbb{F}$-adapted process $\tilde{S}$ is termed the **pre-default value**.
Forward Credit Default Swap

Definition

A forward CDS issued at time $s$, with start date $U$, maturity $T$, and recovery at default is a defaultable claim $(0, A, Z, \tau)$ where

$$
\begin{align*}
dA_t &= -\kappa \mathbb{1}_{[U, T]}(t) \, dL_t, \\
Z_t &= \delta_t \mathbb{1}_{[U, T]}(t).
\end{align*}
$$

- An $\mathcal{F}_s$-measurable rate $\kappa$ is the CDS rate.
- An $\mathbb{F}$-adapted process $L$ specifies the tenor structure of fee payments.
- An $\mathbb{F}$-adapted process $\delta : [U, T] \rightarrow \mathbb{R}$ represents the default protection.

Lemma

The value of the forward CDS equals, for every $t \in [s, U]$,  

$$
S_t(\kappa) = B_t \mathbb{E}_Q \left( \mathbb{1}_{\{U < \tau \leq T\}} B_{\tau}^{-1} Z_\tau \mid \mathcal{G}_t \right) - \kappa B_t \mathbb{E}_Q \left( \int_{t \wedge U}^{\tau \wedge T} B_u^{-1} \, dL_u \mid \mathcal{G}_t \right).
$$

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Credit Default Swaps and Swaptions
Lemma

The value of a credit default swap started at \( s \), equals, for every \( t \in [s, U] \),

\[
S_t(\kappa) = \mathbb{1}_{\{t<\tau\}} \frac{B_t}{G_t} \mathbb{E}_Q \left( - \int_U^T B_u^{-1} \delta_u \, dG_u - \kappa \int_{[U,T]} B_u^{-1} G_u \, dL_u \bigg| \mathcal{F}_t \right).
\]

Note that \( S_t(\kappa) = \mathbb{1}_{\{t<\tau\}} \tilde{S}_t(\kappa) \) where the \( \mathbb{F} \)-adapted process \( \tilde{S}(\kappa) \) is the pre-default value. Moreover

\[
\tilde{S}_t(\kappa) = \tilde{P}(t, U, T) - \kappa \tilde{A}(t, U, T)
\]

where

- \( \tilde{P}(t, U, T) \) is the pre-default value of the protection leg,
- \( \tilde{A}(t, U, T) \) is the pre-default value of the fee leg per one unit of \( \kappa \).
The forward CDS rate is defined similarly as the forward swap rate for a default-free interest rate swap.

**Definition**

The forward market CDS at time $t \in [0, U]$ is the forward CDS in which the $\mathcal{F}_t$-measurable rate $\kappa$ is such that the contract is valueless at time $t$.

The corresponding pre-default forward CDS rate at time $t$ is the unique $\mathcal{F}_t$-measurable random variable $\kappa(t, U, T)$, which solves the equation

$$\widetilde{S}_t(\kappa(t, U, T)) = 0.$$
Lemma

For every $t \in [0, U],$

$$\kappa(t, U, T) = \frac{\tilde{P}(t, U, T)}{\tilde{A}(t, U, T)} = -\frac{\mathbb{E}_Q\left(\int_U^T B_\delta^{-1} dG_u \bigg| \mathcal{F}_t\right)}{\mathbb{E}_Q\left(\int_{[U, T]} B_\delta^{-1} G_u dL_u \bigg| \mathcal{F}_t\right)} = \frac{M_t^P}{M_t^A}$$

where the $(\mathbb{Q}, \mathbb{F})$-martingales $M^P$ and $M^A$ are given by

$$M_t^P = -\mathbb{E}_Q\left(\int_U^T B_\delta^{-1} dG_u \bigg| \mathcal{F}_t\right)$$

and

$$M_t^A = \mathbb{E}_Q\left(\int_{[U, T]} B_\delta^{-1} G_u dL_u \bigg| \mathcal{F}_t\right).$$
Credit Default Swaption: I

**Definition**

A *credit default swaption* is a call option with expiry date $R \leq U$ and zero strike written on the value of the forward CDS issued at time $0 \leq s < R$, with start date $U$, maturity $T$, and an $\mathcal{F}_s$-measurable rate $\kappa$.

The swaption’s payoff $C_R$ at expiry equals $C_R = (S_R(\kappa))^+$. 

**Lemma**

*For a forward CDS with an $\mathcal{F}_s$-measurable rate $\kappa$ we have, for every $t \in [s, U]$,*

$$S_t(\kappa) = 1_{\{t < \tau\}} \tilde{A}(t, U, T)(\kappa(t, U, T) - \kappa).$$

*It is clear that*

$$C_R = 1_{\{R < \tau\}} \tilde{A}(R, U, T)(\kappa(R, U, T) - \kappa)^+. $$

*A credit default swaption is formally equivalent to a call option on the forward CDS rate with strike $\kappa$. This option is knocked out if default occurs prior to $R$. 

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Credit Default Swaps and Swaptions
Credit Default Swaption: II

**Lemma**

The price at time $t \in [s, R]$ of a credit default swaption equals

$$ C_t = \mathbb{1}_{\{t<\tau\}} \frac{B_t}{G_t} \mathbb{E}_Q \left( \frac{G_R}{B_R} \tilde{A}(R, U, T)(\kappa(R, U, T) - \kappa)^+ \mid \mathcal{F}_t \right). $$

Define an equivalent probability measure $\hat{Q}$ on $(\Omega, \mathcal{F}_R)$ by setting

$$ \frac{d\hat{Q}}{dQ} = \frac{M_R^A}{M_0^A}, \quad Q\text{-a.s.} $$

**Proposition**

The price of the credit default swaption equals, for every $t \in [s, R]$,

$$ C_t = \mathbb{1}_{\{t<\tau\}} \tilde{A}(t, U, T) \mathbb{E}_{\hat{Q}} \left( (\kappa(R, U, T) - \kappa)^+ \mid \mathcal{F}_t \right) = \mathbb{1}_{\{t<\tau\}} \tilde{C}_t. $$

The forward CDS rate $(\kappa(t, U, T), t \leq R)$ is a $(\hat{Q}, \mathcal{F})$-martingale.
Let the filtration $\mathbb{F}$ be generated by a Brownian motion $W$ under $\mathbb{Q}$.

Since $M^P$ and $M^A$ are strictly positive $(\mathbb{Q}, \mathbb{F})$-martingales, we have that
\[
dM^P_t = M^P_t \sigma^P_t \, dW_t, \quad dM^A_t = M^A_t \sigma^A_t \, dW_t,
\]
for some $\mathbb{F}$-adapted processes $\sigma^P$ and $\sigma^A$.

**Lemma**

The forward CDS rate $(\kappa(t, U, T), \ t \in [0, R])$ is $(\mathbb{Q}, \mathbb{F})$-martingale and
\[
d\kappa(t, U, T) = \kappa(t, U, T) \sigma^\kappa_t \, d\hat{W}_t
\]
where $\sigma^\kappa = \sigma^P - \sigma^A$ and the $(\mathbb{Q}, \mathbb{F})$-Brownian motion $\hat{W}$ equals
\[
\hat{W}_t = W_t - \int_0^t \sigma^A_u \, du, \quad \forall \ t \in [0, R].
\]
Proposition

Assume that the volatility $\sigma^\kappa = \sigma^P - \sigma^A$ of the forward CDS spread is deterministic. Then the pre-default value of the credit default swaption with strike level $\kappa$ and expiry date $R$ equals, for every $t \in [0, U]$,

$$\tilde{C}_t = \tilde{A}_t \left( \kappa_t N(d_+ (\kappa_t, U - t)) - \kappa N(d_- (\kappa_t, U - t)) \right)$$

where $\kappa_t = \kappa(t, U, T)$ and $\tilde{A}_t = \tilde{A}(t, U, T)$. Equivalently,

$$\tilde{C}_t = \tilde{P}_t N(d_+ (\kappa_t, t, R)) - \kappa \tilde{A}_t N(d_- (\kappa_t, t, R))$$

where $\tilde{P}_t = \tilde{P}(t, U, T)$ and

$$d_\pm (\kappa_t, t, R) = \frac{\ln(\kappa_t/\kappa) \pm \frac{1}{2} \int_t^R (\sigma'^\kappa(u))^2 \, du}{\sqrt{\int_t^R (\sigma'^\kappa(u))^2 \, du}}.$$
Assumption 1

Definition

For any \( u \in \mathbb{R}_+ \), we define the \( \mathbb{F} \)-martingale \( G^u_t = \mathbb{Q}(\tau > u | \mathcal{F}_t) \) for \( t \in [0, T] \).

- Let \( G_t = G^t_t \). Then the process \((G_t, t \in [0, T])\) is an \( \mathbb{F} \)-supermartingale.
- We also assume that \( G \) is a strictly positive process.

Assumption

There exists a family of \( \mathbb{F} \)-adapted processes \((f^x_t; t \in [0, T], x \in \mathbb{R}_+)\) such that, for any \( u \in \mathbb{R}_+ \),

\[
G^u_t = \int_u^\infty f^x_t \, dx, \quad \forall t \in [0, T].
\]
For any fixed \( t \in [0, T] \), the random variable \( f_t \) represents the conditional density of \( \tau \) with respect to the \( \sigma \)-field \( \mathcal{F}_t \), that is,

\[
f_t^x \, dx = \mathbb{Q}(\tau \in dx \mid \mathcal{F}_t).
\]

We write \( f_t^t = f_t \) and we define \( \hat{\lambda}_t = G_t^{-1} f_t \).

**Lemma**

*Under Assumption 1, the process \((M_t, \, t \in [0, T])\) given by the formula*

\[
M_t = H_t - \int_0^t (1 - H_u) \hat{\lambda}_u \, du
\]

*is a \( \mathcal{G} \)-martingale.*

It can be deduced from the lemma that \( \hat{\lambda} = \lambda \) is the default intensity.
Assumption 2

The filtration $\mathbb{F}$ is generated by a one-dimensional Brownian motion $W$.

We now work under Assumptions 1-2. We have that

- For any fixed $u \in \mathbb{R}_+$, the $\mathbb{F}$-martingale $G^u$ satisfies, for $t \in [0, T]$,

$$G^u_t = G^u_0 + \int_0^t g^u_s \, dW_s$$

for some $\mathbb{F}$-predictable, real-valued process $(g^u_t, t \in [0, T])$.

- For any fixed $x \in \mathbb{R}_+$, the process $(f^x_t, t \in [0, T])$ is an $(\mathbb{Q}, \mathbb{F})$-martingale and thus there exists an $\mathbb{F}$-predictable process $(\sigma^x_t, t \in [0, T])$ such that, for $t \in [0, T]$,

$$f^x_t = f^x_0 + \int_0^t \sigma^x_s \, dW_s.$$
The following relationship is valid, for any $u \in \mathbb{R}_+$ and $t \in [0, T]$,

$$g_t^u = \int_u^\infty \sigma_t^x \, dx.$$

By applying the Itô-Wentzell-Kunita formula, we obtain the following auxiliary result, in which we denote $g_s^s = g_s$ and $f_s^s = f_s$.

**Lemma**

*The Doob-Meyer decomposition of the survival process $G$ equals, for every $t \in [0, T]$,*

$$G_t = G_0 + \int_0^t g_s \, dW_s - \int_0^t f_s \, ds.$$

*In particular, $G$ is a continuous process.*
Volatility of Pre-Default Value

Under the assumption that \( B, Z \) and \( A \) are deterministic, the volatility of the pre-default value process can be computed explicitly in terms of \( \sigma_t^u \). Recall that, for \( t \in [0, T] \),

\[
    f_t^x = f_0^x + \int_0^t \sigma_s^x \, dW_s, \quad g_t^u = \int_u^\infty \sigma_t^x \, dx.
\]

**Corollary**

If \( B, Z \) and \( A \) are deterministic then we have that, for every \( t \in [0, T] \),

\[
    d\tilde{S}_t = \left( (r(t) + \lambda_t)\tilde{S}_t - \lambda_t Z(t) \right) \, dt + dA(t) + \zeta_t^T \, dW_t
\]

with \( \zeta_t^T = G_t^{-1} B(t) \nu_t^T \) where

\[
    \nu_t^T = B^{-1}(T) XG_t^T + \int_t^T B^{-1}(u)Z(u)\sigma_u^u \, du + \int_t^T B^{-1}(u)g_t^u \, dA(u).
\]
Lemma

If $B$, $\delta$ and $L$ are deterministic then the forward CDS rate satisfies under $\hat{Q}$

$$d\kappa(t, U, T) = \kappa(t, U, T)(\sigma_t^P - \sigma_t^A) \, d\hat{W}_t$$

where the process $\hat{W}$, given by the formula

$$\hat{W}_t = W_t - \int_0^t \sigma_u^A \, du, \quad \forall \, t \in [0, R],$$

is a Brownian motion under $\hat{Q}$ and

$$\sigma_t^P = \left( \int_U B^{-1}(u) \delta(u) \sigma_t^u \, du \right) \left( \int_U B^{-1}(u) \delta(u) f_t^u \, du \right)^{-1}$$

$$\sigma_t^A = \left( \int_U B^{-1}(u) g_t^u \, du \right) \left( \int_U B^{-1}(u) G_t^u \, du \right)^{-1}.$$
We make the following standing assumptions:

1. The default intensity process $\lambda$ is governed by the CIR dynamics

$$d\lambda_t = \mu(\lambda_t) \, dt + \nu(\lambda_t) \, dW_t$$

where $\mu(\lambda) = a - b\lambda$ and $\nu(\lambda) = c\sqrt{\lambda}$.

2. The default time $\tau$ is given by

$$\tau = \inf \left\{ t \in \mathbb{R}_+ : \int_0^t \lambda_u \, du \geq \Theta \right\}$$

where $\Theta$ is a random variable with the unit exponential distribution, independent of the filtration $\mathcal{F}$.
From the martingale property of $f^u$ we have, for every $t \leq u$,

$$f_t^u = \mathbb{E}_Q(f_u | \mathcal{F}_t) = \mathbb{E}_Q(\lambda_u G_u | \mathcal{F}_t).$$

The immersion property holds between $\mathbb{F}$ and $\mathbb{G}$ so that $G_t = \exp(-\Lambda_t)$, where $\Lambda_t = \int_0^t \lambda_u \, du$ is the hazard process. Therefore

$$f_t^s = \mathbb{E}_Q(\lambda_s e^{-\Lambda_s} | \mathcal{F}_t).$$

Let us denote

$$H_t^s = \mathbb{E}_Q(e^{-(\Lambda_s - \Lambda_t)} | \mathcal{F}_t) = \frac{G_t^s}{G_t}.$$ 

It is important to note that for the CIR model

$$H_t^s = e^{m(t,s)-n(t,s)\lambda_t} = \hat{H}(\lambda_t, t, s)$$

where $\hat{H}(\cdot, t, s)$ is a strictly decreasing function when $t < s$. 
We assume that:

1. The tenor structure process $L$ is deterministic.
2. The savings account is $B$ is deterministic. We denote $\beta = B^{-1}$.
3. We also assume that $\delta$ is constant.

**Proposition**

The volatility of the forward CDS rate satisfies $\sigma^k = \sigma^P - \sigma^A$ where

$$
\sigma^P_t = \nu(\lambda_t) \frac{\beta(T)H^T_t n(t, T) - \beta(U)H^U_t n(t, U) + \int_U^T r(u)\beta(u)H^U_t n(t, u) \, du}{\beta(U)H^U_t - \beta(T)H^T_t - \int_U^T r(u)\beta(u)H^U_t \, du}
$$

and

$$
\sigma^A_t = \nu(\lambda_t) \frac{\int_{[U, T]} \beta(u)H^U_t n(t, u) \, dL(u)}{\int_{[U, T]} \beta(u)H^U_t \, dL(u)}.
$$
One can show that

\[ C_R = \mathbb{1}_{\{R < \tau\}} \left( \delta \int_0^T B(R, u) \lambda^R_u \, du - \kappa \int_{[0,T]} B(R, u) H^u_R \, dL(u) \right) ^+. \]

Straightforward computations lead to the following representation

\[ C_R = \mathbb{1}_{\{R < \tau\}} \left( \delta B(R, U) H^U_R - \int_{[0,T]} B(R, u) H^u_R \, d\chi(u) \right) ^+. \]

where the function \( \chi : \mathbb{R}^+ \to \mathbb{R} \) satisfies

\[ d\chi(u) = -\delta \frac{\partial \ln B(R, u)}{\partial u} \, du + \kappa \, dL(u) + \delta \, d1_{[T, \infty]}(u). \]
Auxiliary Functions

- We define auxiliary functions $\zeta : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ and $\psi : \mathbb{R} \rightarrow \mathbb{R}^+$ by setting

$$
\zeta(x) = \delta B(R, U) \hat{H}(x, R, U)
$$

and

$$
\psi(y) = \int_{[U, \tau]} B(R, u) \hat{H}(y, R, u) \, d\chi(u).
$$

- There exists a unique $\mathcal{F}_R$-measurable random variable $\lambda_R^*$ such that

$$
\zeta(\lambda_R) = \delta B(R, U) \hat{H}(\lambda_R, R, U) = \int_{[U, \tau]} B(R, u) \hat{H}(\lambda_R^*, R, u) \, d\chi(u) = \psi(\lambda_R^*).
$$

- It suffices to check that $\lambda_R^* = \psi^{-1}(\zeta(\lambda_R))$ is the unique solution to this equation.
The payoff of the credit default swaption admits the following representation

\[ C_R = \mathbb{1}_{\{ R < \tau \}} \int_{[u, T]} B(R, u) (\hat{H}(\lambda^*_R, R, u) - \hat{H}(\lambda_R, R, u))^+ d\chi(u). \]

Let \( D^0(t, u) \) be the price at time \( t \) of a unit defaultable zero-coupon bond with zero recovery maturing at \( u \geq t \) and let \( B(t, u) \) be the price at time \( t \) of a (default-free) unit discount bond maturing at \( u \geq t \).

If the interest rate process \( r \) is independent of the default intensity \( \lambda \) then \( D^0(t, u) \) is given by the following formula

\[ D^0(t, u) = \mathbb{1}_{\{ t < \tau \}} B(t, u) H^u_t. \]
Let $P(\lambda_t, U, u, K)$ stand for the price at time $t$ of a put bond option with strike $K$ and expiry $U$ written on a zero-coupon bond maturing at $u$ computed in the CIR model with the interest rate modeled by $\lambda$.

**Proposition**

Assume that $R = U$. Then the payoff of the credit default swaption equals

$$C_U = \int_{]U,T]} (K(u)D^0(U,U) - D^0(U,u))^+ d\chi(u)$$

where $K(u) = B(U,u)\hat{H}(\lambda^*_U, U, u)$ is deterministic, since $\lambda^*_U = \psi^{-1}(\delta)$.

The pre-default value of the credit default swaption equals

$$\tilde{C}_t = \int_{]U,T]} B(t,u)P(\lambda_t, U, u, \hat{K}(u)) d\chi(u)$$

where $\hat{K}(u) = K(u)/B(U,u) = \hat{H}(\lambda^*_U, U, u)$. 
The price $P^u_t := P(\lambda_t, U, u, \hat{K}(u))$ of the put bond option in the CIR model with the interest rate $\lambda$ is known to be

$$P^u_t = \hat{K}(u) H^U_t \mathbb{P}^U(H^U_u \leq \hat{K}(u) | \lambda_t) - H^u_t \mathbb{P}^u(H^u_u \leq \hat{K}(u) | \lambda_t)$$

where $H^u_t = \hat{H}(\lambda_t, t, u)$ is the price at time $t$ of a zero-coupon bond maturing at $u$.

Let us denote $Z_t = H^u_t / H^U_t$ and let us set, for every $u \in [U, T]$,

$$\mathbb{P}^u(H^u_u \leq \hat{K}(u) | \lambda_t) = \psi_u(t, Z_t).$$

Then the pricing formula for the bond put option becomes

$$P^u_t = \hat{K}(u) H^U_t \psi_U(t, Z_t) - H^u_t \psi_u(t, Z_t)$$
Trading Strategies

- Let \( \varphi = (\varphi^1, \varphi^2) \) be a trading strategy, where \( \varphi^1 \) and \( \varphi^2 \) are \( \mathbb{G} \)-adapted processes.
- The wealth of \( \varphi \) equals, for every \( t \in [s, R] \),
  \[
  V_t(\varphi) = \varphi^1_t S_t(\kappa) + \varphi^2_t A(t, U, T)
  \]
  and thus the pre-default wealth satisfies, for every \( t \in [s, R] \),
  \[
  \tilde{V}_t(\varphi) = \varphi^1_t \tilde{S}_t(\kappa) + \varphi^2_t \tilde{A}(t, U, T).
  \]
- It is enough to search for \( \mathbb{F} \)-adapted processes \( \tilde{\varphi}^i, i = 1, 2 \) such that the equality
  \[
  \mathbbm{1}_{\{t < \tau\}} \varphi^i_t = \tilde{\varphi}^i_t
  \]
  holds for every \( t \in [s, R] \).
We have the following

**Proposition**

The hedging strategy \( \tilde{\varphi} = (\tilde{\varphi}^1, \tilde{\varphi}^2) \) for the credit default swaption equals, for \( t \in [s, U] \),

\[
\begin{align*}
\tilde{\varphi}^1_t &= \frac{\tilde{\xi}_t}{\kappa(t, U, T)\sigma_t^\kappa}, \\
\tilde{\varphi}^2_t &= \frac{\tilde{C}_t - \tilde{\varphi}^1_t \tilde{S}_t(\kappa)}{\tilde{A}(t, U, T)}
\end{align*}
\]

where \( \tilde{\xi} \) is the process satisfying

\[
\frac{\tilde{C}_U}{\tilde{A}(U, U, T)} = \frac{\tilde{C}_0}{\tilde{A}(0, U, T)} + \int_0^U \tilde{\xi}_t \, d\tilde{W}_t.
\]

All terms were already computed, except for the process \( \tilde{\xi} \).
In view of the general representation for the hedging strategy when $\mathbb{F}$ is the Brownian filtration, we are searching for the process $\tilde{\xi}$ such that

$$d(\tilde{C}_t/\tilde{A}(t, U, T)) = \tilde{\xi}_t d\tilde{W}_t.$$ 

**Proposition**

Assume that $R = U$. Then we have that, for every $t \in [0, U]$,

$$\tilde{\xi}_t = \frac{1}{\tilde{A}_t} \left( \int_{[U, T]} B(t, u) \left( \vartheta_t H_t^u (b_t^u - b_t^U) - P_t^u b_t^U \right) d\chi(u) - \tilde{C}_t \sigma_t^A \right)$$

where

$$\tilde{A}_t = \tilde{A}(t, U, T), \quad H_t^u = \hat{H}(\lambda_t, t, u), \quad b_t^u = cn(t, u) \sqrt{\lambda_t}, \quad P_t^u = P(\lambda_t, U, u, \hat{K}(u))$$

and

$$\vartheta_t = \hat{K}(u) \frac{\partial \Psi_u}{\partial Z}(t, Z_t) - \Psi_u(t, Z_t) - Z_t \frac{\partial \Psi_u}{\partial Z}(t, Z_t).$$
Proposition

Consider the CIR default intensity model with a deterministic short-term interest rate. The replicating strategy \( \tilde{\varphi} = (\tilde{\varphi}_1, \tilde{\varphi}_2) \) for the credit default swaption maturing at \( R = U \) equals, for any \( t \in [0, U] \),

\[
\begin{align*}
\tilde{\varphi}_1^t &= \frac{\tilde{\xi}_t}{\kappa(t, U, T)\sigma_t^\kappa}, \\
\tilde{\varphi}_2^t &= \frac{\tilde{C}_t - \tilde{\varphi}_1^t \tilde{S}_t(\kappa)}{\tilde{A}(t, U, T)},
\end{align*}
\]

where the processes \( \sigma^\kappa, \tilde{C} \) and \( \tilde{\xi} \) are given in previous results.

Note that for \( R < U \) the problem remains open, since a closed-form solution for the process \( \tilde{\xi} \) is not readily available in this case.